NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

DEPARTMENT OF MECHANICAL ENGINEERING

COURSE MATERIALS

MAT 201 PARTIAL DIFFERENTIAL EQUATIONS AND COMPLEX ANALYSIS

VISION OF THE INSTITUTION

To mould true citizens who are millennium leaders and catalysts of change through excellence in education.

MISSION OF THE INSTITUTION

NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.

We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2002
- Course offered : B.Tech in Mechanical Engineering

  M.Tech in Machine Design
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of Dr. A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Producing internationally competitive Mechanical Engineers with social responsibilities and sustainable employability through viable strategies as well as competent exposure oriented quality education.

DEPARTMENT MISSION

M1	Imparting high impact education by providing conductive teaching learning environment.
M2	Fostering effective modes of continuous learning process with moral and ethical values.
M3	Enhancing leadership qualities with social commitment, professional attitude, unity, team spirit and communication skill.
M4	Introducing present scenario in research and development through collaborative efforts blended with industry and institution.

PROGRAMME EDUCATIONAL OBJECTIVES

PEO No.	Program Educational Objectives Statements
PEO1	Graduates shall have strong practical and technical exposures in the field of Mechanical Engineering and will contribute to the society through Innovation and Enterprise.
PEO2	Graduates will have the demonstrated ability to analyze, formulate and solve design engineering/thermal engineering/materials and manufacturing/design issues and real life problems.
PEO3	Graduates will be capable of pursuing Mechanical Engineering profession with good communication skills, leadership qualities, team spirit and professional ethics.
PEO4	Graduates will sustain an appetite for continuous learning by pursuing higher education and research in the allied areas of technology.

PROGRAM OUTCOMES (POS)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a

member or leader in diverse teams, and in multidisciplinary settings.

10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSO)

PSO1	Graduates able to apply principles of engineering, basic sciences and analytics including multi-variant calculus and higher order partial differential equations.
PSO2	Graduates able to perform modelling, analysing, designing and simulating physical systems, components and processes.
PSO3	Graduates able to work professionally on mechanical systems, thermal systems and production systems.

COURSE OUTCOMES

CO 1	Understand the concept and the solution of partial differential equation.
CO 2	Analyse and solve one dimensional wave equation and heat equation.
CO 3	Understand complex functions, its continuity differentiability with the use of Cauchy-Riemann equations.
CO 4	Evaluate complex integrals using Cauchy's integral theorem and Cauchy's integral formula, understand the series expansion of analytic function
CO 5	Understand the series expansion of complex function about a singularity and Apply residue theorem to compute several kinds of real integrals.

CO VS PO'S AND PSO'S MAPPING

CO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO 1	3	3	3	3	2	1	-	-	-	2	-	2
CO 2	3	3	3	3	2	1	-	-	-	2	-	2
CO 3	3	3	3	3	2	1	-	-	-	2	-	2
CO 4	3	3	3	3	2	1	-	-	-	2	-	2
CO 5	3	3	3	3	2	1	-	-	-	2	-	2

CO	PSO1	PSO2	PSO3
CO1	2	1	
CO2	2	1	
CO3	1	1	
CO4	1	1	
CO5	1	1	

Note: H-Highly correlated=3, M-Medium correlated=2, L-Less correlated=1

SYLLABUS

Module 1 (Partial Differential Equations) (8 hours)

(Text 1-Relevant portions of sections 17.1, 17.2, 17.3, 17.4, 17.5, 17.7, 18.1, 18.2)

Partial differential equations, Formation of partial differential equations –elimination of arbitrary constants-elimination of arbitrary functions, Solutions of a partial differential equations, Equations solvable by direct integration, Linear equations of the first order-Lagrange's linear equation, Non-linear equations of the first order -Charpit's method, Solution of equation by method of separation of variables.

Module 2 (Applications of Partial Differential Equations) (10 hours)
(Text 1-Relevant portions of sections 18.3,18.4, 18.5)

One dimensional wave equation- vibrations of a stretched string, derivation, solution of the wave equation using method of separation of variables, D'Alembert's solution of the wave equation, One dimensional heat equation, derivation, solution of the heat equation

Module 3 (Complex Variable – Differentiation) (9 hours)
(Text 2: Relevant portions of sections13.3, 13.4, 17.1, 17.2 , 17.4)

Complex function, limit, continuity, derivative, analytic functions, Cauchy-Riemann equations, harmonic functions, finding harmonic conjugate, Conformal mappings- mappings $W= Z^2$, $W=e^Z$ .Linear fractional transformation W = 1/Z fixed points, Transformation W=Sin Z

Module 4 (Complex Variable – Integration) (9 hours)
(Text 2- Relevant topics from sections14.1, 14.2, 14.3, 14.4,15.4**)**

Complex integration, Line integrals in the complex plane, Basic properties, First evaluation method-indefinite integration and substitution of limit, second evaluation method-use of a representation of a path, Contour integrals, Cauchy integral theorem (without proof) on simply connected domain,Cauchy integral theorem (without proof) on multiply connected domain Cauchy Integral formula (without proof), Cauchy Integral formula for derivatives of an analytic function, Taylor's series and Maclaurin series.

Module 5 (Complex Variable – Residue Integration) (**9 hours**)
(Text 2- Relevant topics from sections16.1, 16.2, 16.3, 16.4 **)**

Laurent's series(without proof), zeros of analytic functions, singularities, poles, removable singularities, essential singularities, Residues, Cauchy Residue theorem (without proof), Evaluation of definite integral using residue theorem, Residue integration of real integrals – integrals of rational functions of Cos θ and Sinθ , integrals of improper integrals of the form $\int_{-\infty}^{\infty} f(x)dx$ with no poles on the real axis.($\int_A^B f(x)dx$ whose integrand become infinite at a point in the interval of integration is excluded from the syllabus), Textbooks:

1. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 44th Edition, 2018.
2. Erwin Kreyszig, Advanced Engineering Mathematics, 10th Edition, John Wiley & Sons, 2016.

References:

1. Peter V. O'Neil, Advanced Engineering Mathematics, Cengage, 7th Edition, 2012

QUESTION BANK

MODULE 1

Q.NO	QUESTIONS	CO	KL	PAGE NO
1	Derive a partial differential equation from the relation $z = f(x + at) + g(x - at)$	CO1	K1	12
2.	Derive a partial differential equation from the relation $z = yf(x) + xg(y)$	CO1	K3	12
3	Find the differential equation of all planes which are at a constant distance a from the origin	CO1	K1	13
4	Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + sin(2x - y) = 0$	CO1	K3	14
5	Use Charpit's methods to solve $q + xp = p^2$	CO1	K3	16
6	Use Charpit's methods to solve $(p^2 + q^2)y = qz$	CO1	K3	18
7	Solve $x(y - z)p + y(z - x)q = z(x - y)$	CO1	K2	19
8	Solve $(y - z)p + (x - y)q = z - x$	CO1	K2	20
9	Solve by the method of separation of variables $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$	CO1	K2	22
10	Using the method of separation of variables, solve $x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = \mathbf{0}$	CO1	K3	24
11	Using the method of separation of variables, solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x, 0) = 6\, e^{-3x}$	CO1	K3	25
12	Solve $xydx + y^2dy = zxy - 2x^2$	CO1	K3	28

MODULE 2

Q.NO	QUESTIONS	CO	KL	PAGE NO
1	Derive One dimensional wave equation	CO2	K2	30
2.	Derive the solution of one dimensional wave equation	CO2	K2	31
3	A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 sin^3 \frac{\pi x}{l}$ Find the displacement of the string at any time.	CO2	K3	32
4	A tightly stretched string with fixed end points $x = 0 \; and \; x = l$ is initially in a position given by $y = y_0 sin^3 \frac{\pi x}{l}$. If it is released from rest from this position find the displacement $y(x, t)$	CO2	K3	34
5	A transversely vibrating string of length 'a' is stretched between two points A and B. The initial displacement of each point of the string is zero and the initial velocity at a distance x from A is kx(a-x). Find the form of string at any subsequent time.	CO2	K3	35
6	Derive Solution of one dimensional wave equation using D Alembert's method	CO2	K1	39
7	Derive One dimensional heat equation	CO2	K1	40
8	Derive Solution of one dimensional heat equation using variable Separable method	CO2	K2	41
9	Find the temperature $U(x, t)$ of a homogeneous bar of heat conducting length l whose end points are kept at zero temperature and whose initi is given by $\frac{ax(l-x)}{l^2}$	CO2	K2	43
10	A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is $u(x, 0) = f(x) = \begin{cases} x \quad , & 0 < x < 50 \\ 100 - x, & 50 < x < 100 \end{cases}$ Find the temperature (x, t) at any time	CO2	K3	44
11	A homogeneous rod of conducting material of length 10 cm has its ends kept at zero temperature and the temperature initially is $u(x, 0) = f(x) = \begin{cases} x \quad , & 0 < x < 5 \\ 10 - x, & 50 < x < 10 \end{cases}$	CO2	K3	45

	Find the temperature (x, t) at any time.			
12	A tightly stretched homogenous string of length 20cm with its fixed ends executes transverse vibrations. Motion starts with zero initial velocity by displacing the string into the form $f(x) = K(x^2 - x^3).\ Find\ the\ deflection\ u(x, t)\ at\ any\ time\ t$	CO2	K3	48

MODULE 3

Q.NO	QUESTIONS	CO	KL	PAGE NO
1	Check whether the function $f(z) = \frac{Re(z^2)}{\lvert z\rvert}$ is continuous at $z = 0$ given $f(0) = 0$	CO3	K2	50
2.	Prove that the function f(x, y) = $x^3 - 3xy^2 - 5y$ is harmonic everywhere. Find its harmonic conjugate.	CO3	K3	52
3	Show that $f(z) = e^z$ is analytic for all z. Find its derivative.	CO3	K1	54
4	If the $function\ u = ax^3 + bxy$ is harmonic then find a and b. Also find its harmonic conjugate.	CO3	K3	55
5	Verify $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function $v\ of\ u$ is no where analytic	CO3	K3	52
6	Find the conjugate function V and express $u + iv$ as an analytic function of z.	CO3	K1	56
7	Show that the function $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic.	CO3	K2	53
8	Find the image of the regions 2<|z|<3 and |argz|<$\frac{\pi}{4}$ under the transformation $w = z^2$ and plot it	CO3	K3	60
9	Find the fixed points of the bilinear transformation $w = \frac{z-1}{z+1}$	CO3	K2	62
10	Find the image of the following infinite strips under the mapping $w = \frac{1}{z}$ $\frac{1}{4} < y < \frac{1}{2}$	CO3	K2	65
11	Find the image of the region $\lvert z-\frac{1}{3}\rvert \leq \frac{1}{3}$ under the transformation w=$\frac{1}{z}$	CO3	K1	70
12	Prove that $f(z) = e^z$ is conformal	CO3	K2	74

MODULE 4

Q.NO	QUESTIONS	CO	KL	PAGE NO
1	Evaluate $\int_C Re\, z\, dz$, C is the shortest path from $1+i \; to \; 3+3i$	CO4	K1	78
2.	Evaluate $\int_C Im\, z^2\, dz$ counter clockwise around the triangle with vertices 0,1, i	CO4	K2	79
3	Evaluate $\int_0^{1+i} (x^2 - iy)$ dz along $y = x$	CO4	K3	81
4	Evaluate $\oint_C \frac{dz}{z-3i}$ C is the circle $\|z\| = \pi$ counter clock wise	CO4	K3	84
5	Evaluate $\oint_C \frac{\sin z}{z+2iz}\, dz \;\; C\colon \|z-4-2i\| = 5.5$	CO4	K1	85
6	Evaluate $\oint_C \frac{e^z}{ze^z - 2iz}\, dz \;\; C\colon \|z\| = 0.6$	CO4	K2	87
7	Integrate$\oint_C \frac{z^6}{(2z-1)^6}\, dz \;\; where\ Cis\ the\ unit\ circle$	CO4	K3	89
8	Integrate $\oint_C \frac{z^3+\sin z}{(z-i)^3}\, dz \;\; where\ Cis\ the\ boundary\ \ of\ a\ square\ wit$ $\pm 2, \pm 2i\ counterclock\ wise$	CO4	K2	90
9	Integrate $\oint_C \frac{\cos\pi z^2 + sin\pi z^2}{(z-1)(z-2)}\, dz \;\; where\ C : \|z\| =$ $3\ clock\ wise$	CO4	K2	93
10	Integrate $\oint_C \frac{\exp(z^2)}{z(z-2i)^2}\, dz \;\; where\ C : \|z-3i\| =$ $2\ clock\ wise$	CO4	K3	95
11	Find the Taylor series $f(z) = \frac{1}{z^2-z-6}\ about\ z = -1$	CO4	K3	96
12	Find the Taylor series of $f(z) = \frac{1}{z}\ about\ z = 2$	CO4	K3	96

MODULE 5

Q.NO	QUESTIONS	CO	KL	PAGE NO
1	1. Expand $f(z) = \frac{1}{z-z^3}$ in Laurent series for the region $1 < \|z+1\| < 2$	CO5	K1	98
2.	Expand $f(z) = \frac{z}{(z+1)(z+2)}$ in Laurent series about z=-2	CO5	K1	99
3	Find the Laurent series of $\frac{1}{z^3 - z^4}$ with Centre 0	CO5	K2	100
4	What type of singularity have the function $f(z) = \frac{1}{\cos z - sin z}$	CO5	K3	103
5	Expand $f(z) = \frac{z-1}{z^2-5z+6}$ $in\ 2 < \|z\| < 3\ as\ a\ Laurent\ series$	CO5	K1	105
6	Determine and classify the singularities of the function $f(z) = e^{\frac{1}{z}}$	CO5	K3	108
7	Find all singular points and corresponding residues of $f(z) = \frac{z+2}{(z+1)^2(z-2)}$	CO5	K3	110
8	Find the residues of $f(z) = \frac{50z}{z^3+2z^2-7z+4}$	CO5	K3	115
9	Find the residue of $\frac{e^z}{z^3}$at its pole.	CO5	K2	113
10	Evaluate$\oint_C \frac{dz}{(z^2+4)^2}$ $\quad whereC: \|z-2-i\| = 3.2$	CO5	K3	116
11	Use residue theorem to evaluate $\int_C \frac{\cos h\ \pi z}{z^2+4}$dz where C is $\|z\|$ = 3.	CO5	K3	117
12	Evaluate $\oint_C \frac{z-23}{z^2-4z-5}\, dz$ $\quad whereC: \|z-i\| = 2$	CO5	K3	118

MODULE I

PARTIAL DIFFERENTIAL EQUATIONS

SECTION:17.1 PARTIAL DIFFERENTIAL EQUATION

An equation that contatins partial derivatives of an unknown function is called a partial differential equation. In a pde the unknown function or dependent variable, say U depends on two or more independent variables.

The following notations are adopted throughout the study of Pde's

p=$\frac{\partial z}{\partial x} = z_x$ q=$\frac{\partial z}{\partial y}$=$z_y$ r=$\frac{\partial^2 z}{\partial x^2} = z_{xx}$ s=$\frac{\partial^2 z}{\partial x \partial y}$ =z_{xy} t=$\frac{\partial^2 z}{\partial y^2}$=$z_{yy}$

SECTION:17.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATION

Pde's are formed by eliminating arbitrary constants or arbitrary functions from a relation which contains three or more variables.

ELIMINATION OF ARBITRARY CONSTANTS

Suppose we have an equation $f(x, y, z, a, b) = 0$ where 'a' and 'b' are arbitrary constants. Let us consider z as a function(dependent variable) of two independent varibles x and y. We now form a pde by eliminating 'a' and 'b' by differentiating the given equation.We get another function $\phi(x, y, z, p, q) = 0$ which is a pde of first order.

REMARK

If the number of arbitrary constants to be eliminated is equal to the number of independent variables then we get a first order pde.

If the number of arbitrary constants to be eliminated is more than the number of independent variables then we get a higher order pde.

Similarly we can eliminate arbitrary functions. Elimination of arbitrary functions forms the PDE $Pp + Qq = R$ $where\ P, Q, R\ are\ functions\ of\ x, y, z$

PROBLEMS

I Find the partial differential equation by elimination arbitrary constants from the following

1.

$z = (x - a)^2 + (y - b)^2$

Solution:

$z = (x - a)^2 + (y - b)^2$---------------------------**(1)**

Differentiating (1) partially w.r.t. x we get

$\frac{\partial z}{\partial x} = 2(x - a)$ **ie,** $p = 2(x - a)$ → $x - a = \frac{p}{2}$

Differentiating (1) partially w.r.t. y we get

$\frac{\partial z}{\partial y} = 2(y - b)$ **ie,** $q = 2(y - b)$ → $y - b = \frac{q}{2}$

$$(1) ===> z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$ie,\quad 4z = p^2 + q^2$ **which is the required PDE**

2.

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

Solution. Differentiating (*i*) partially with respect to *x* and *y*, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \quad \text{or} \quad \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$$

and

$$\frac{2\partial z}{\partial y} = \frac{2y}{b^2} \quad \text{or} \quad \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$$

Substituting these values of $1/a^2$ and $1/b^2$ in (*i*), we get

$$2z = xp + yq$$

as the desired partial differential equation of the first order.

3. $z = ax + by + a^2 + b^2$

Solution:

$$\frac{\partial z}{\partial x} = a \quad or\ p = a$$

$$\frac{\partial z}{\partial y} = b \quad or \quad q = b$$

Substituting in the given equation we get

$$z = px + qy + p^2 + q^2$$

4. Find the differential equation of all planes which are at a constant distance a from the origin

Solution:

Equation of a plane in normal form is

$$lx + my + nz = a$$

Where l,m,n are the d.c.s of the normal from the origin to the plane.

Then $\qquad l^2 + m^2 + n^2 = 1$ or $n = \sqrt{(1 - l^2 - m^2)}$

$\therefore$ (*i*) becomes $\qquad lx + my + \sqrt{(1 - l^2 - m^2)}\ z = a$

Differentiating partially w.r.t. x, we get

$$l + \sqrt{(1 - l^2 - m^2)}\ .\ p = 0$$

Differentiating partially w.r.t. y, we get

$$m + \sqrt{(1 - l^2 - m^2)}\ .\ q = 0$$

Now we have to eliminate l, m from (*ii*), (*iii*) and (*iv*).

From (*iii*), $l = -\sqrt{(1 - l^2 - m^2)}\ .\ p$ and $m = -\sqrt{(1 - l^2 - m^2)}\ .\ q$

Squaring and adding, $l^2 + m^2 = (1 - l^2 - m^2)(p^2 + q^2)$

or $\qquad (l^2 + m^2)(1 + p^2 + q^2) = p^2 + q^2$ or $1 - l^2 - m^2 = 1 - \dfrac{p^2 + q^2}{1 + p^2 + q^2} = \dfrac{1}{1 + p^2 + q^2}$

Also $\qquad l = -\dfrac{p}{\sqrt{(1 + p^2 + q^2)}}$ and $m = -\dfrac{q}{\sqrt{(1 + p^2 + q^2)}}$

Substituting the values of l, m and $1 - l^2 - m^2$ in (*ii*), we obtain

$$\frac{-px}{\sqrt{(1 + p^2 + q^2)}} - \frac{qy}{\sqrt{(1 + p^2 + q^2)}} + \frac{1}{\sqrt{(1 + p^2 + q^2)}}\ z = a$$

or $\qquad z = px + qy + a\sqrt{(1 + p^2 + q^2)}$ which is the required partial differential equation.

5.

Find the differential equation of all spheres of fixed radius having their centres in the xy plane

Solution:

Equarion of sphere with centre(h,k,0) in xy plane and radius 'r' is

$(\boldsymbol{x - h})^2 + (\boldsymbol{y - k})^2 + \boldsymbol{z}^2 = \boldsymbol{r}^2$

Differentiating the given equation w.r.t 'x'

$$2(x - h) + 2z\frac{\partial z}{\partial x} = 0 \quad ie, \quad x - h + pz = 0$$

$$x - h = -pz$$

Differentiating the given equation w.r.t 'y'

$$2(y - k) + 2z\frac{\partial z}{\partial y} = 0 \quad ie, \quad y - k + qz = 0$$

$$y - k = -qz$$

Therefore the given equation become $(-pz)^2 + (-qz)^2 + z^2 = r^2$

$$z^2(p^2+q^2+1)=r^2$$

6. Form a pde by eliminating a,b,c $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solution:

Consider the function $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-----------(1)$

Differentiating (1) partially w.r.t x we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0$$

$$\frac{x}{a^2}=-\frac{z}{c^2}\frac{\partial z}{\partial x}-----------(2)$$

$$\frac{c^2}{a^2}=-\frac{z}{x}p-----------(3)$$

Differentiating (2) partially w.r.t x we get

$$\frac{1}{a^2}=-\frac{1}{c^2}[z\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}*\frac{\partial z}{\partial x}]$$

$$\frac{c^2}{a^2}=-[zr+p^2]-----------(4)$$

Equating (3) and (4) we get

$$-\frac{z}{x}p=-[zr+p^2]$$

$$zr+p^2-\frac{z}{x}p=0 \text{ which is the required pde}$$

II Form pde by eliminating arbitrary function

1. $xyz=\phi(x+y+z)$

Solution:

Solution is of the form $\boxed{Pp+Qq=R}$

Where $P=\begin{vmatrix}U_y & U_z\\ V_y & V_z\end{vmatrix}$ $Q=\begin{vmatrix}U_z & U_x\\ V_z & V_x\end{vmatrix}$ $R=\begin{vmatrix}U_x & U_y\\ V_x & V_y\end{vmatrix}$

$$xyz = \phi(x+y+z) \Rightarrow f(xyz, x+y+z) = 0$$

Here U=xyz, V=x+y+z

P=$\begin{vmatrix} xz & xy \\ 1 & 1 \end{vmatrix}$**=x(z-y)** **Q =**$\begin{vmatrix} xy & yz \\ 1 & 1 \end{vmatrix}$**=y(x-z)** **R=**$\begin{vmatrix} yz & xz \\ 1 & 1 \end{vmatrix}$**=z(y-x)**

Solution Pp+Qq=R→ x(z-y)p+ y(x-z)q= z(y-x)

2. $z = y^2 + 2f\left(\frac{1}{x} + logy\right)$

   Solution:

   Solution is of the form

Where P=$\begin{vmatrix} U_y & U_z \\ V_y & V_z \end{vmatrix}$ **Q=**$\begin{vmatrix} U_z & U_x \\ V_z & V_x \end{vmatrix}$ $\boxed{\text{Pp+Qq=R}}$ **R=**$\begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix}$

Here $z = y^2 + 2f\left(\frac{1}{x} + logy\right) \rightarrow \phi\left(\frac{z-y^2}{2}, \frac{1}{x} + logy\right) = 0$

Here U=$\frac{z-y^2}{2}$ **V=**$\frac{1}{x} + logy$

P=$\begin{vmatrix} -y & \frac{1}{2} \\ \frac{1}{y} & 0 \end{vmatrix} = -\frac{1}{2y}$ **Q=**$\begin{vmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{x^2} \end{vmatrix} = -\frac{1}{2x^2}$ **R=**$\begin{vmatrix} 0 & -y \\ -\frac{1}{x^2} & \frac{1}{y} \end{vmatrix} = -\frac{y}{x^2}$

Therefore solution is $-\frac{1}{2y}p + -\frac{1}{2x^2}q = -\frac{y}{x^2}$ → $x^2p + yq - 2y^2 = 0$

3. $f(x+y+z, x^2+y^2+z^2) = 0$

   Solution:

   Here U= $x+y+z$ **V=**$x^2+y^2+z^2$
   P=$\begin{vmatrix} 1 & 1 \\ 2y & 2z \end{vmatrix}$**=2(z-y)** **Q=**$\begin{vmatrix} 1 & 1 \\ 2z & 2x \end{vmatrix}$**=2(x-z)** **R=**$\begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = 2(y-x)$

   Solution is 2(z-y)p+2(x-z)q= $2(y-x)$ **→ (z-y)p+(x-z)q=**$(y-x)$

4. **Form pde by eliminating arbitrary function**

   (a)$z = (x+y)\phi(x^2-y^2)$ **(b)**$z = f(x+at) + g(x-at)$

Solution:

$$p = \frac{\partial z}{\partial x} = (x+y)\,\phi'(x^2-y^2)\,.\,2x + \phi(x^2-y^2),$$

$$q = \frac{\partial z}{\partial y} = (x+y)\,\phi'(x^2-y^2)\,.\,(-2y) + \phi(x^2-y^2)$$

From (*i*), $$p - \frac{z}{x+y} = 2x\,(x+y)\,\phi'(x^2-y^2)$$

From (*ii*), $$q - \frac{z}{x+y} = -2y\,(x+y)\,\phi'(x^2-y^2)$$

Division gives $$\frac{p - z/(x+y)}{q - z/(x+y)} = -\frac{x}{y}$$

i.e., $$[p(x+y)-z]y + [q(x+y)-z]x$$

i.e., $$(x+y)(py+qx) - z(x+y) = 0$$

Hence $py + qz = z$ is required equation.

(*b*) We have $z = f(x+at) + g(x-at)$

Differentiating z partially with respect to x and t,

$$\frac{\partial z}{\partial x} = f'(x+at) + g'(x-at),\quad \frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at)$$

$$\frac{\partial z}{\partial t} = af'(x+at) - ag'(x-at),\quad \frac{\partial^2 z}{\partial t^2} = a^2 f''(x+at) + a^2 g''(x-at) = a^2\frac{\partial^2 z}{\partial x^2}$$

Thus the desired partial differential equation is $\dfrac{\partial^2 z}{\partial t^2} = a^2\dfrac{\partial^2 z}{\partial x^2}$

which is an equation of the second order and (*i*) is its solution.

SECTION:17.3 SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

It is clear from the above examples that a partial differential equation can result both from elimination of arbitrary constants and from the elimination of arbitrary functions.

The solution $$f(x, y, z, a, b) = 0 \qquad ...(1)$$
of a first order partial differential equation which contains two arbitrary constants is called a *complete integral*.

A solution obtained from the complete integral by assigning particular values to the arbitrary constants is called a particular integral.

If we put $b = \phi(a)$ in (1) and find the envelope of the family of surfaces $f[x, y, z, \phi(a)] = 0$, then we get a solution containing an arbitrary function ϕ, which is called the *general integral*.

The envelope of the family of surfaces (1), with parameters a and b, if it exists, is called a *singular integral*. The singular integral differs from the particular integral in that it is not obtained from the complete integral by giving particular values to the constants.

SECTION:17.4 EQUATIONS SOLVABLE BY DIRECT INTEGRATION

We now consider such partial differential equations which can be solved by direct integration. In place of the usual constants of integration, we must, however use arbitrary functions of variables held fixed.

PROBLEMS

1. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + sin(2x - y) = 0$

Solution. Integrating twice with respect to x (keeping y fixed),

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x - y) = f(y)$$

$$\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x - y) = xf(y) + g(y).$$

Now integrating with respect to y (keeping x fixed)

$$z + x^3y^3 - \frac{1}{4}\cos(2x - y) = x\int f(y)dy + \int g(y)dy + w(x)$$

The result may be simplified by writing

$$\int f(y)dy = u(y) \text{ and } \int g(y)dy = v(y).$$

Thus $z = \frac{1}{4}\cos(2x - y) - x^3y^3 + xu(y) + v(y) + w(x)$ where u, v, w are arbitrary functions.

2. Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ *given that when* $x = 0$ $z = e^y$ *and* $\frac{\partial z}{\partial x} = 1$

Solution. If z were function of x alone, the solution would have been $z = A\sin x + B\cos x$, where A and B are constants. Since z is a function of x and y, A and B can be arbitrary functions of y. Hence the solution of the given equation is $z = f(y)\sin x + \phi(y)\cos x$

$$\therefore \quad \frac{\partial z}{\partial x} = f(y)\cos x - \phi(y)\sin x$$

When $x = 0; z = e^y$, $\quad \therefore e^y = \phi(y)$. $\quad$ When $x = 0$, $\frac{\partial z}{\partial x} = 1$, $\quad \therefore \; 1 = f(y)$.

Hence the desired solution is $z = \sin x + e^y \cos x$.

3. Solve $\frac{\partial^2 z}{\partial x \partial y} = sinxsiny$ for which $\frac{\partial z}{\partial y} = -2siny$ $when\ x = 0$, $and\ z = 0$ when y is an odd multiple of $\frac{\Pi}{2}$

Solution. Given equation is $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

Integrating w.r.t. x, keeping y constant, we get

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y)$$

When $x = 0$, $\frac{\partial z}{\partial y} = -2 \sin y$, $\quad \therefore \quad -2 \sin y = -\sin y + f(y)$ or $f(y) = -\sin y$

$\therefore$ (*i*) becomes $\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$

Now integrating w.r.t. y, keeping x constant, we get

$$z = \cos x \cos y + \cos y + g(x) \qquad ...(ii)$$

When y is an odd multiple of $\pi/2$, $z = 0$.

$\therefore \quad 0 = 0 + 0 + g(x)$ or $g(x) = 0 \qquad [\because \cos(2n+1)\pi/2 = 0]$

Hence from (*ii*), the complete solution is $z = (1 + \cos x)\cos y$.

SECTION:17.5 LINEAR EQUATIONS OF FIRST ORDER

Consider a PDE which is linear in P,Q,R is of the form Pp+Qq=R , where P,Q,R are the functions of x,y,z. This is called Lagrange's linear equation which is of order one.

Method for solving Lagrange's linear equation.

1. **Form the equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. This is known as Lagrange's auxiliary equation or subsidiary equation.**
2. **By the method of grouping or by the method of multipliers or both solve the auxiliary equations to get two independent solutions U(x,y,z)=C_1 , V(x,y,z)=C_2**

Method of grouping

Suppose that one of the variable is either absent or cancels out from any pair of fractions of equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ and then a solutions can be obtained by using ususal methods. The same procedure is repeated with another pair of fractions of equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ for second independent solutions.

Method of multiplier

If l,m,n are three multipliers, then by a well known principles of algebra, each fraction $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ equal to $\frac{ldx+mdy+ndz}{lP+mQ+nR}$. Choose l,m,n such that lP+mQ+nR=0 then ldx+mdy+ndz=0. Integrating we get U(x,y,z)=C_1. This method may be repeated to get another independent solution V(x,y,z)= C_2. This multiplierr l,m,n are called Lagrangian Multiplier

3. **General solution is ϕ(U,V)=0 or U=ϕ(V)**

PROBLEMS

1. **Solve** $\frac{y^2 z}{x}p+xzq=y^2$

Solution. Rewriting the given equation as

$$y^2zp+x^2zq=y^2x,$$

The subsidiary equations are $\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{y^2x}$

The first two fractions give $x^2dx=y^2dy$.

Integrating, we get $x^3-y^3=a$

Again the first and third fractions give $xdx=zdz$

Integrating, we get $x^2-z^2=b$

Hence from (*i*) and (*ii*), the complete solution is

$$x^3-y^3=f(x^2-z^2).$$

2. **Solve** $pz-qz=z^2+(x+y)^2$

Solution: Here the auxiliary equations are

$\frac{dx}{z}=\frac{dy}{-z}=\frac{dz}{z^2+(x+y)^2}$---------------------(1)

The first two fractions we get

dx=-dy

integrating x=-y+c➔ x+y=c=➔

$\boxed{U=x+y}$

Again first and third

$\frac{dx}{z}=\frac{dz}{z^2+(x+y)^2}$➔$\frac{dx}{z}=\frac{dz}{z^2+c^2}$

Integrating $x=\frac{1}{2}log(z^2+c^2)+c_1$➔ 2x- $log(z^2+c^2)$=$2c_1$

$\boxed{V=2x-log(z^2+c^2)}$

The general solution is ϕ(x+y, 2x- $log(z^2+c^2))=0$

3. **Solve** $xydx+y^2dy=zxy-2x^2$

Solution:

A.E is $\frac{dx}{xy}=\frac{dy}{y^2}=\frac{dz}{zxy-2x^2}$

From first two equations $\frac{dx}{xy} = \frac{dy}{y^2}$ ➔ $\frac{dx}{x} = \frac{dy}{y}$

Integrating logx=logy+logc_1 ➔ $\frac{x}{y} = c_1$---(1)

$\boxed{U=\frac{x}{y}}$

From 2nd and 3rd $\frac{dy}{y^2} = \frac{dz}{zxy-2x^2}$

Substitute x=$c_1 y$ (from eqn (1))

$$\frac{dy}{y^2} = \frac{dz}{zc_1yy - 2(c_1y)^2}$$

$$\frac{dy}{y^2} = \frac{dz}{c_1y^2(z-2c_1)}$$

$$c_1\, dy = \frac{dz}{z-2c_1}$$

Integrating $c_1\, y = log(z-2c_1) + log\ c_2$

X=log(z-2$\frac{x}{y}$)+ $log\ c_2$ ➔ log e^x- log(z-2$\frac{x}{y}$)= $log\ c_2$

$\frac{e^x}{z-2\frac{x}{y}}$=$c_2$ ➔ $\frac{ye^x}{yz-2x} = c_2$

$\boxed{V=\frac{ye^x}{yz-2x}}$

General solution $\phi(\frac{x}{y}, \frac{ye^x}{yz-2x}) = 0$

4. **Solve** $p - 2q = 3x^2 sin(y+2x)$

   Solution: Here the auxiliary equations are

$\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 sin(y+2x)}$---------------------(1)

From first two equations $\frac{dx}{1} = \frac{dy}{-2}$ ➔ 2 dx=-dy

Integrating 2x+y=c_1

$\boxed{U=2x+y}$

From 1st and 3rd

$\frac{dx}{1} = \frac{dz}{3x^2 sin(y+2x)}$ ➔ dx=$\frac{dz}{3x^2 sin(c_1)}$

ie, $3x^2 sinc_1\ \ dx = dz$

$Integrating\ x^3 sinc_1 = z + c_2$ ➔ $x^3 sinc_1 - z = c_2$

$\boxed{V=x^3 sinc_1 - z}$

General solution $\phi(2x + y, x^3 sinc_1 - z) = 0$

5. Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using the multipliers $1/x$, $1/y$ and $1/z$, we have

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$\therefore$ $$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \text{ which on integration gives}$$

$$\log x + \log y + \log z = \log a \qquad \text{or} \qquad xyz = a$$

Therefore $\boxed{U = xyz}$

Using the multipliers $\frac{1}{x^2}, \frac{1}{y^2}$ and $\frac{1}{z^2}$, we get

$$\text{each fraction} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$\therefore$ $$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0, \text{ which on integrating gives}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$$

$\boxed{V = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$

General solution $\phi\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$

6. Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \qquad \ldots(i)$$

$$\text{Each of these equations} = \frac{dx - dy}{x^2 - y^2 - (y-x)z} = \frac{dy - dz}{y^2 - z^2 - x(z-y)}$$

i.e., $$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)} \qquad \text{or} \qquad \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating, $$\log(x-y) = \log(y-z) + \log c \qquad \text{or} \qquad \frac{x-y}{y-z} = c \qquad \ldots(ii)$$

$$\text{Each of the subsidiary equations } (i) = \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{xdx + ydy + zdz}{(x+y+z)(x^2 + y^2 + z^2 - yz - zx - xy)} \qquad \ldots(iii)$$

$$\text{Also each of the subsidiary equations} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy} \qquad \ldots(iv)$$

Equating (*iii*) and (*iv*) and cancelling the common factor, we get

$$\frac{xdx + ydy + zdz}{x + y + z} = dx + dy + dz$$

or $$\int (xdx + ydy + zdz) = \int (x + y + z)d\,(x + y + z) + c'$$

or $$x^2 + y^2 + z^2 = (x + y + z)^2 + 2c' \quad \text{or} \quad xy + yz + zx + c' = 0 \qquad \ldots(v)$$

Combining (*ii*) and (*v*), the general solution is

$$\phi\left(\frac{x-y}{y-z}, xy + yz + zx\right) = 0$$

7. Solve $(mz - ny)p + (nx - lz)q = ly - mx$

Solution. Here the subsidiary equations are $\dfrac{dx}{mz - ny} = \dfrac{dy}{mx - lz} = \dfrac{dz}{ly - mx}$

Using multipliers x, y, and z, we get each fraction $= \dfrac{xdx + ydy + zdz}{0}$

$\therefore$ $xdx + ydy + zdz = 0$ which on integration gives $x^2 + y^2 + z^2 = a$

Again using multipliers l, m and n, we get each fraction $= \dfrac{ldx + mdy + ndz}{0}$

$\therefore$ $ldx + mdy + ndx = 0$ which on integration gives $lx + my + nz = b$

General solution $\phi(x^2 + y^2 + z^2, lx + my + nz) = 0$

8. **Solve $(x^2 - y^2 - z^2)p + 2xy\,q = 2xz$**

Solution. Here the subsidiary equations are $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$

From the last two fractions, we have $\dfrac{dy}{y} = \dfrac{dz}{z}$

which on integration gives $\log y = \log z + \log a$ or $y/z = a$

Using multipliers x, y and z, we have

each fraction $= \dfrac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$ $\quad\therefore\quad \dfrac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \dfrac{dz}{z}$

which on integration gives $\log(x^2 + y^2 + z^2) = \log z + \log b$

or $$\frac{x^2 + y^2 + z^2}{z} = b$$

General solution $\phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0$

9. Solve $(y - z)p + (x - y)q = z - x$

Solution

Here the auxiliary equations are $\frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x}$

Choose 1,1,1 as multiplier, each fraction is equal to $\frac{dx+dy+dz}{0}$

Integrating $x + y + z = a$

$\boxed{U=x+y+z}$

Choose x,z,y as multiplier, each fraction is equal to $\frac{xdx+zdy+ydz}{0}$

⇨ $xdx + d(zy) = 0$

Integrating $\frac{x^2}{2} + zy = b$ **General solution** $\phi(x + y + z, \frac{x^2}{2} + zy) = 0$

SECTION:17.7 NONLINEAR EQUATIONS OF FIRST ORDER

Those equations in which p and q occur other than in the first degree are called *non-linear partial differential equations of the first order*. The *complete solution* of such an equation contains only two arbitrary constants (*i.e.*, equal to the number of independent variables involved) and the particular integral is obtained by giving particular values to the constants.]

CHARPIT'S METHOD

We now explain a general method for finding the complete integral of a non-linear partial differential equation which is due to Charpit.

Consider the equation

$$f(x, y, z, p, q) = 0 \quad \text{...(1)}$$

Since z depends on x and y, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = pdx + qdy \quad \text{...(2)}$$

Now if we can find another relation involving x, y, z, p, q such as $\phi(x, y, z, p, q) = 0$ ...(3)

then we can solve (1) and (3) for p and q and substitute in (2). This will give the solution provided (2) is integrable.

To determine ϕ, we differentiate (1) and (3) with respect to x and y giving

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \quad \text{...(4)}$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x} = 0 \quad \text{...(5)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \quad \text{...(6)}$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} q + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial y} = 0 \quad \text{...(7)}$$

Eliminating $\frac{\partial p}{\partial x}$ between the equations (4) and (5), we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial p}-\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial p}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial p}-\frac{\partial \phi}{\partial z}\frac{\partial f}{\partial p}\right)p+\left(\frac{\partial f}{\partial q}\frac{\partial \phi}{\partial p}-\frac{\partial \phi}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x}=0 \quad ...(8)$$

Also eliminating $\frac{\partial q}{\partial y}$ between the equations (6) and (7), we obtain

$$\left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial q}-\frac{\partial \phi}{\partial y}\frac{\partial f}{\partial q}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial q}-\frac{\partial \phi}{\partial z}\frac{\partial f}{\partial q}\right)q+\left(\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial q}-\frac{\partial \phi}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y}=0 \quad ...(9)$$

Adding (8) and (9) and using $\frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial p}{\partial y}$,

we find that the last terms in both cancel and the other terms, on rearrangement, give

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y}=0 \quad ...(10)$$

i.e.,

$$\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z}+\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q}=0 \quad ...(11)$$

This is Lagrange's linear equation (§ 17.5) with x, y, z, p, q as independent variables and ϕ as the dependent variable. Its solution will depend on the solution of the subsidiary equations

$$\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{\partial \phi}{0}$$

An integral of these equations involving p or q or both, can be taken as the required relation (3), which alongwith (1) will give the values of p and q to make (2) integrable. Of course, we should take the simplest of the integrals so that it may be easier to solve for p and q.

PROBLEMS

1. **Solve $(p^2+q^2)y=qz$**

Solution. Let $f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{-2py}=\frac{dy}{z-2qy}=\frac{dz}{-qz}=\frac{dp}{-pq}=\frac{dq}{p^2}$$

The last two of these give $pdp + qdq = 0$

Integrating, $p^2 + q^2 = c^2$ ...(*ii*)

Now to solve (*i*) and (*ii*), put $p^2 + q^2 = c^2$ in (*i*), so that $q = c^2y/z$

Substituting this value of q in (*ii*), we get $p = c\sqrt{(z^2-c^2y^2)}/z$

Hence $$dz = pdx + qdy = \frac{c}{z}\sqrt{(z^2-c^2y^2)}\,dx+\frac{c^2y}{z}dy$$

or $$zdz - c^2y\,dy = c\sqrt{(z^2-c^2y^2)}\,dx \quad \text{or} \quad \frac{\frac{1}{2}d(z^2-c^2y^2)}{\sqrt{(z^2-c^2y^2)}}=c\,dx$$

Integrating, we get $\sqrt{(z^2-c^2y^2)} = cx + a$ or $z^2 = (a + cx)^2 + c^2y^2$ which is the required complete integral.

2. **Solve $2xz - px^2 - 2qxy + pq = 0$**

Solution. Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0$ ...(i)

Charpit's subsidiary equations are

$$\frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dp}{2z - 2qy} = \frac{dq}{0}$$

$\therefore$ $dq = 0$ or $q = a$.

Putting $q = a$ in (i), we get $p = \dfrac{2x(z - ay)}{x^2 - a}$

$\therefore$ $dz = pdx + qdy = \dfrac{2x(z - ay)}{x^2 - a}dx + ady$ or $\dfrac{dz - ady}{z - ay} = \dfrac{2x}{x^2 - a}dx$

Integrating, $\log(z - ay) = \log(x^2 - a) + \log b$

or $z - ay = b(x^2 - a)$ or $z = ay + b(x^2 - a)$

which is the required complete solution.

3. **Solve $2z + p^2 + qy + 2y^2 = 0$**

Solution. Let $f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2$

Charpit's subsidiary equations are

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2 + qy)} = \frac{dp}{2p} = \frac{dq}{4y + 3q}$$

From first and fourth ratios,

$$dp = -dx \quad \text{or} \quad p = -x + a$$

Substituting $p = a - x$ in the given equation, we get

$$q = \frac{1}{y}[-2z - 2y^2 - (a - x)^2]$$

$\therefore$ $dz = pdx + qdy = (a - x)dx - \dfrac{1}{y}[2z + 2y^2 + (a - x)^2]dy$

Multiplying both sides by $2y^2$,

$$2y^2dz + 4yz\,dy = 2y^2(a - x)dx - 4y^3dy - 2y(a - x)^2dy$$

Integrating $2zy^2 = -[y^2(a - x)^2 + y^4] + b$

or $y^2[(x - a)^2 + 2z + y^2] = b$, which is the desired solution.

APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS

SECTION:18.1 INTRODUCTION

In physical problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions. The differential equation together with these boundary conditions, constitute a *boundary value problem.*

In problems involving ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions which are difficult to adjust so as to satisfy the given boundary conditions. Most of the boundary value problems involving linear partial differential equations can be solved by the following method.

SECTION:18.2 METHOD OF SEPERATION OF VARIABLES

It involves a solution which breaks up into a product of functions each of which contains only one of the variables.

PROBLEMS

1. **Solve by the method of separation of variables** $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

Solution. Assume the trial solution $z = X(x)Y(y)$ ...(*i*)

where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation, we have

$$X''Y - 2X'Y + XY' = 0 \quad \text{where } X' = \frac{dX}{dx}, Y' = \frac{dY}{dy} \text{ etc.}$$

Separating the variables, we get $\frac{X'' - 2X'}{X} = -\frac{Y'}{Y}$ ...(*ii*)

Since x and y are independent variables, therefore, (*ii*) can only be true if each side is equal to the same constant, a (say).

$$\therefore \quad \frac{X'' - 2X'}{X} = a, \text{ i.e. } X'' - 2X' - aX = 0 \quad \ldots(iii)$$

and $$-Y'/Y = a, \quad \text{i.e.,} \quad Y' + aY = 0 \quad \ldots(iv)$$

To solve the ordinary linear equation (*iii*), the auxiliary equation is

$$m^2 - 2m - a = 0, \quad \text{whence } m = 1 \sqrt{(1 + a)}.$$

$\therefore$ the solution of (*iii*) is $X = c_1 e^{[1 + \sqrt{(1+a)}]x} + c_2 e^{[1 - \sqrt{(1+a)}]x}$

and the solution of (*iv*) is $Y = c_3 e^{-ay}$.

Substituting these values of X and Y in (*i*), we get

$$z = \{c_1 e^{[1+\sqrt{(1+a)}]x} + c_2 e^{[1-\sqrt{(1+a)}]x}\} \cdot c_3 e^{-ay}$$

i.e., $$z = \{k_1 e^{[1+\sqrt{(1+a)}]x} + k_2 e^{[1-\sqrt{(1+a)}]x}\} e^{-ay}$$

which is the required complete solution.

2. Solve by the method of separation of variables

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u, \; where \qquad u(x,0) = 6e^{-3x}$$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u - -(i)$$

Solution. Assume the solution $u(x, t) = X(x)T(t)$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \text{or} \quad (X' - X)T = 2XT'$$

or $$\frac{X' - X}{2X} = \frac{T'}{T} = k \text{ (say)}$$

$$\therefore \quad X' - X - 2kX = 0 \quad \text{or} \quad \frac{X'}{X} = 1 + 2k \quad \ldots(i) \quad \text{and} \quad \frac{T'}{T} = k \quad \ldots(ii)$$

Solving (*i*), $\log X = (1 + 2k)x + \log c$ or $X = ce^{(1+2k)x}$

From (*ii*), $\log T = kt + \log c'$ or $T = c'e^{kt}$

Thus $u(x, t) = XT = cc'\, e^{(1+2k)x}\, e^{kt}$ ...(*iii*)

Now $6e^{-3x} = u(x, 0) = cc'\, e^{(1+2k)x}$

$\therefore$ $cc' = 6$ and $1 + 2k = -3$ or $k = -2$

Substituting these values in (*iii*), we get

$$u = 6e^{-3x} e^{-2t} \quad \text{i.e.,} \quad u = 6e^{-(3x+2t)}$$ which is the required solution.

3. Solve by the method of separation of variables

$$x\,\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$$

Solution: $x\,\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$ **--------------(1)**

$U(x,y) = X(x)Y(y)$ **where X function of x and Y function of y only**

$\frac{\partial u}{\partial x} = X'Y$--------------(2)

$\frac{\partial u}{\partial y} = XY' ----(3)$

$substituting\ (2) and\ (3) in\ (1)$ we get

$x(X'Y) - 2y(XY') = 0$

$x(X'Y)$= $2y(XY')$

$\frac{xX'}{X} = \frac{2yY'}{Y} = k$

$\frac{xX'}{X} = k$ ➔ $\frac{X'}{X} = \frac{k}{x}$ ➔ $\frac{dX}{X} = \frac{k}{x}\ dx$

$integrating$

$\log X = k\log x + \log c_1$ ➔ $\log X = \log c_1 x^k$ ➔ $X = c_1 x^k$

$\frac{2yY'}{Y} = k$ ➔ $\frac{Y'}{Y} = \frac{k}{2y}$ ➔ $\frac{dY}{Y} = \frac{k}{2y}\ dy$

$integrating$

$\log Y = \frac{k}{2}\log y + \log c_2$ ➔ $\log Y = \log c_2 y^{\frac{k}{2}}$ ➔ $Y = c_2 y^{\frac{k}{2}}$

General solution u(x,y)= $c_1 c_2 x^k y^{\frac{k}{2}}$=$cx^k y^{\frac{k}{2}}$

MODULE 2

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

18.3 PARTIAL DIFFERENTIAL EQUATIONS OF ENGINEERING

A number of problems in engineering give rise to the following well-known partial differential equations :

(*i*) *Wave equation* : $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$.

(*ii*) *One dimensional heat flow equation* : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

(*iii*) *Two dimensional heat flow equation* which in steady state becomes the two dimensional *Laplace's equation* : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(*iv*) *Transmission line equations.*

(*v*) *Vibrating membrane.* Two dimensional wave equation.

(*vi*) *Laplace's equation* in three dimensions.

Besides these, the partial differential equations frequently occur in the theory of Elasticity and Hydraulics.

Starting with the method of separation of variables, we find their solutions subject to specific boundary conditions and the combination of such solution gives the desired solution. Quite often a certain condition is not applicable. In such cases, the most general solution is written as the sum of the particular solutions already found and the constants are determined using Fourier series so as to satisfy the remaining conditions.

18.4 VIBRATIONS OF A STRETCHED STRING—WAVE EQUATION

Consider a tightly stretched elastic string of length l and fixed ends A and B and subjected to constant tension T (Fig. 18.1). The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.

Let the string be released from rest and allowed to vibrate. We shall study the subsequent motion of the string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB, of the string entirely in one plane.

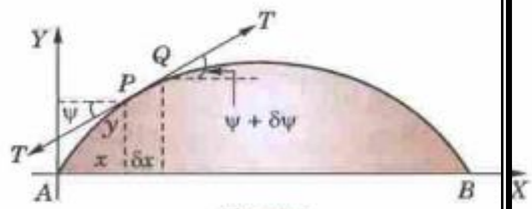


Fig. 18.1

Taking the end A as the origin, AB as the x-axis and AY perpendicular to it as the y-axis ; so that the motion takes place entirely in the xy-plane. Figure 18.1 shows the string in the position APB at time t. Consider the motion of the element PQ of the string between its points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$, where the tangents make angles ψ and $\psi + \delta\psi$ with the x-axis. Clearly the element is moving upwards with the acceleration $\partial^2 y/\partial t^2$. Also the vertical component of the force acting on this element.

$$= T \sin(\psi + \delta\psi) - T \sin \psi = T[\sin(\psi + \delta\psi) - \sin \psi]$$

$$= T\,[\tan(\psi + \delta\psi) - \tan\psi], \text{ since } \psi \text{ is small } = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}\right]$$

If m be the mass per unit length of the string, then by Newton's second law of motion, we have

$$m\delta x \cdot \frac{\partial^2 y}{\partial t^2} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}\right] \quad i.e., \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}}{\delta x}\right]$$

Taking limits as $Q \to P$ *i.e.*, $dx \to 0$, we have $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$, where $c^2 = \dfrac{T}{m}$ ...(1)

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional *wave equation*.

(2) Solution of the wave equation. Assume that a solution of (1) is of the form

$z = X(x)T(t)$ where X is a function of x and T is a function of t only.

Then $\dfrac{\partial^2 y}{\partial t^2} = X \cdot T''$ and $\dfrac{\partial^2 y}{\partial x^2} = X'' \cdot T$

Substituting these in (1), we get $XT'' = c^2X''T$ *i.e.*, $\dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T''}{T}$ ...(2)

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations :

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{...(3)} \quad \text{and} \quad \frac{d^2T}{dt^2} - kc^2T = 0 \quad \text{...(4)}$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say $X = c_1e^{px} + c_2e^{-px}$; $T = c_3e^{cpt} + c_4e^{-cpt}$.

(*ii*) *When k is negative* and $= -p^2$ say $X = c_5 \cos px + c_6 \sin px$; $T = c_7 \cos cpt + c_8 \sin cpt$.

(*iii*) *When k is zero.* $X = c_9x + c_{10}$; $T = c_{11}t + c_{12}$.

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-cpt}) \quad \text{...(5)}$$

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \quad \text{...(6)}$$

$$y = (c_9x + c_{10})(c_{11}t + c_{12}) \quad \text{...(7)}$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by (6), *i.e.*, of the form

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{...(8)}$$

is the only suitable solution of one dimensional wave equation.

Example 18.3. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form y = a sin (πx/l) from which it is released at time t = 0. Show that the displacement of any point at a distance x from one end at time t is given by*

$$y(x, t) = a \sin(\pi x/l) \cos(\pi ct/l).$$ (V.T.U., 2010 ; S.V.T.U., 2008 ; Kerala, 2005 ; U.P.T.U., 2004)

Solution. The vibration of the string is given by $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ...(*i*)

As the end points of the string are fixed, for all time,

$y(0, t) = 0$ ...(*ii*) and $y(l, t) = 0$ ...(*iii*)

Since the initial transverse velocity of any point of the string is zero,

therefore, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ ...(*iv*)

Also $y(x, 0) = a \sin(\pi x/l)$ ...(*v*)

Now we have to solve (*i*) subject to the *boundary conditions* (*ii*) and (*iii*) and *initial conditions* (*iv*) and (*v*). Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$ ...(*vi*)

By (*ii*), $y(0, t) = C_1(C_3 \cos cpt + C_4 \sin cpt) = 0$

For this to be true for all time, $C_1 = 0$.

Hence $y(x, t) = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt)$ ...(*vii*)

and $\frac{\partial y}{\partial t} = C_2 \sin px \{C_3(-cp \,.\, \sin cpt) + C_4(cp \,.\, \cos cpt)\}$

$\therefore$ By (*iv*), $\left(\frac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \,.\, (C_4\, cp) = 0$, whence $C_2 C_4 cp = 0$.

If $C_2 = 0$, (*vii*) will lead to the trivial solution $y(x, t) = 0$,

$\therefore$ the only possibility is that $C_4 = 0$.

Thus (*vii*) becomes $y(x, t) = C_2 C_3 \sin px \cos cpt$ ...(*viii*)

$\therefore$ By (*iii*), $y(l, t) = C_2 C_3 \sin pl \cos cpt = 0$ for all t.

Since C_2 and $C_3 \neq 0$, we have $\sin pl = 0$. $\therefore$ $pl = n\pi$, *i.e.*, $p = n\pi/l$, where n is an integer.

Hence (*i*) reduces to $y(x, t) = C_2 C_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$.

[These are the solutions of (*i*) satisfying the boundary conditions. These functions are called the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$ of the vibrating string. The set of values $\lambda_1, \lambda_2, \lambda_3, \ldots$ is called its **spectrum**.]

Finally, imposing the last condition (*v*), we have $y(x, 0) = C_2 C_3 \sin \frac{n\pi x}{l} = a \sin \frac{\pi x}{l}$

which will be satisfied by taking $C_2 C_3 = a$ and $n = 1$.

Hence the required solution is $y(x, t) = a \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}$ ...(*x*)

Example 18.4. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 (\pi x/l)$. If it is released from rest from this position, find the displacement $y(x, t)$.*

(*Rajasthan, 2006 ; V.T.U., 2003 ; J.N.T.U., 2002*)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(*ii*

Also the initial conditions are $y(x, 0) = y_0 \sin^3 \left(\dfrac{\pi x}{l}\right)$ ...(*iii*

and $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$ ...(*iv*

Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (*ii*), $y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

$\therefore$ $y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$

Also by (*ii*), $y(l, t) = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $$y(x, t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{cn\pi t}{l} + c_4 \sin \frac{cn\pi t}{l}\right) \quad ...(v$$

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l}\right)\frac{cn\pi}{l}\left(-c_3 \sin \frac{cn\pi t}{l} + c_4 \cos \frac{cn\pi t}{l}\right)$$

By (*iv*), $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \dfrac{n\pi x}{l}\right)\dfrac{cn\pi}{l} \,.\, c_4 = 0$, *i.e.* $c_4 = 0$.

Thus (*v*) becomes $y(x, t) = c_2 c_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l} = b_n \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$

Adding all such solutions the general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad ...(vi)$$

$\therefore$ from (*iii*), $$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

or $$y_0 \left\{ \frac{3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4} \right\} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \ldots$$

Comparing both sides, we have

$$b_1 = 3\, y_0/4,\ b_2 = 0,\ b_3 = -y_0/4,\ b_4 = b_5 = \ldots = 0.$$

Hence from (*vi*), the desired solution is

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}.$$

Example 18.5. *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at any time $t > 0$.*

(*Bhopal, 2008 ; Madras, 2006 ; J.N.T.U., 2005 ; P.T.U., 2005*)

Solution. The equation of the string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $y(0, t) = 0,\ y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = \mu x(l - x)$ ...(*iii*)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad \ldots(iv)$$

The solution of (*i*) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)\,(c_3 \cos cpt + c_4 \sin cpt)$$

By (*ii*), $y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

$\therefore$ $y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$

Also by (*ii*) $y(l, t) = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $$y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l} \right) \qquad \ldots(v)$$

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l} \right) \frac{n\pi c}{l} \left(-c_3 \sin \frac{n\pi ct}{l} + c_4 \cos \frac{n\pi ct}{l} \right)$$

$\therefore$ by (*iv*) $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \frac{n\pi x}{l} \right) \frac{n\pi c}{l} \cdot c_4 = 0$$

Thus (*v*) becomes $$y(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

Adding all such solutions, the general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad \text{...(vi)}$$

From (*iii*), $$\mu(lx - x^2) = y(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l}\int_0^l \mu(lx - x^2)\sin\frac{n\pi x}{l}\,dx, \text{ by Fourier half-range sine series}$$

$$= \frac{2\mu}{l}\left\{\left|(lx - x^2)\left(-\frac{\cos n\pi x/l}{n\pi/l}\right)\right|_0^l - \int_0^l (l - 2x)\left(-\frac{\cos n\pi x/l}{n\pi/l}\right)dx\right\}$$

$$= \frac{2\mu}{l}\cdot\frac{1}{n\pi}\left\{\int_0^l (l - 2x)\frac{\cos n\pi x}{l}\,dx\right\} = \frac{2\mu}{n\pi}\left\{\left|(l - 2x)\frac{\sin n\pi x/l}{n\pi/l}\right|_0^l - \int_0^l (-2)\frac{\sin n\pi x/l}{n\pi/l}\,dx\right\}$$

$$= \frac{2\mu}{n\pi}\cdot\frac{2l}{n\pi}\int_0^l \sin\frac{n\pi x}{l}\,dx = \frac{4\mu l}{n^2\pi^2}\left|\frac{-\cos n\pi x/l}{n\pi/l}\right|_0^l = \frac{4\mu l^2}{n^3\pi^3}\{1 - (-1)^n\}$$

Hence from (*vi*), the desired solution is

$$y(x, t) = \frac{4\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1 - (-1)^n}{n^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

$$= \frac{8\mu l^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi}{l}x\cos\frac{(2m-1)\pi ct}{l}.$$

Example 18.6. *A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 \sin^3 \pi x/l$. Find the displacement $y(x, t)$.*

(S.V.T.U., 2008 ; V.T.U., 2008 ; U.P.T.U., 2006)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $y(0, t) = 0$, $y(l, t) = 0$ ...(ii)

Also the initial conditions are $y(x, 0) = 0$ ...(iii)

and
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l} \quad \text{...(iv)}$$

Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (ii), $\quad y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time $c_1 = 0$.

$\therefore \quad y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$

Also $\quad y(l, t) = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $\quad pl = n\pi \quad$ or $\quad p = \dfrac{n\pi}{l}$, n being an integer.

Thus
$$y(x, t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{cn\pi}{l} t + c_4 \sin \frac{cn\pi}{l} t\right)$$

By (iii), $\quad 0 = c_2 c_3 \sin \dfrac{n\pi x}{l} \quad$ for all x *i.e.*, $c_2 c_3 = 0$

$\therefore \quad y(x, t) = b_n \sin \dfrac{n\pi x}{l} \sin \dfrac{cn\pi t}{l} \quad$ where $b_n = c_2 c_4$

Adding all such solutions, the general solution of (i) is

$$y(x, t) = \sum b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l} \quad \text{...(v)}$$

Now
$$\frac{\partial y}{\partial t} = \sum b_n \sin \frac{n\pi x}{l} \cdot \frac{cn\pi}{l} \cos \frac{cn\pi t}{l}$$

By (iv), $\quad v_0 \sin^3 \dfrac{\pi x}{l} = \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \sum \dfrac{cn\pi}{l} b_n \sin \dfrac{n\pi x}{l}$

or
$$\frac{v_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right) = \sum \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l} \qquad [\because \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta]$$

$$= \frac{c\pi}{l} b_1 \sin \frac{\pi x}{l} + \frac{2c\pi}{l} b_2 \sin \frac{2\pi x}{l} + \frac{3c\pi}{l} b_3 \sin \frac{3\pi x}{l} + \dots$$

Equating coefficients from both sides, we get

$$\frac{3v_0}{4} = \frac{c\pi}{l} b_1, \quad 0 = \frac{2c\pi}{l} b_2, \quad -\frac{v_0}{4} = \frac{3c\pi}{l} b_3, \ldots$$

$\therefore$ $$b_1 = \frac{3lv_0}{4c\pi}, \quad b_3 = -\frac{lv_0}{12c\pi}, \quad b_2 = b_4 = b_3 = \ldots = 0$$

Substituting in (v), the desired solution is

$$y = \frac{lv_0}{12c\pi}\left(9 \sin \frac{\pi x}{l} \sin \frac{c\pi t}{l} - \sin \frac{3\pi x}{l} \sin \frac{3c\pi t}{l}\right).$$

Example 18.7. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.* (Anna, 2009 ; U.P.T.U., 2002)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(ii)

Also the initial conditions are $y(x, 0) = 0$ ...(iii)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l - x) \qquad \ldots(iv)$$

As in example 18.6, the general solution of (i) satisfying the conditions (ii) and (iii) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} . \sin \frac{n\pi ct}{l} \qquad \ldots(v)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} . \cos \frac{n\pi ct}{l} . \left(\frac{n\pi c}{l}\right)$$

By (iv), $$\lambda x(l - x) = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{\pi c}{l} \sum_{n=1}^{\infty} nb_n \sin \frac{n\pi x}{l}$$

$\therefore$ $$\frac{\pi c}{l} nb_n = \frac{2}{l} \int_0^l \lambda x(l - x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2\lambda}{l} \left| (lx - x^2)\left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3}(1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3} [1 - (-1)^n]$$

or $$b_n = \frac{4\lambda l^3}{c\pi^4 n^4}[1 - (-1)^n] = \frac{8\lambda l^3}{c\pi^4 (2m - 1)^4} \text{ taking } n = 2m - 1.$$

Hence, from (v), the desired solution is

$$y = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m - 1)^4} \sin \frac{(2m - 1)\pi x}{l} \sin \frac{(2m - 1)\pi ct}{l}.$$

Example 18.8. *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.*

(*Kerala, 2005*)

Solution. Let B and C be the points of the trisection of the string $OA(=l)$ (Fig. 18.2). Initially the string is held in the form $OB'C'A$, where $BB' = CC' = a$ (say).

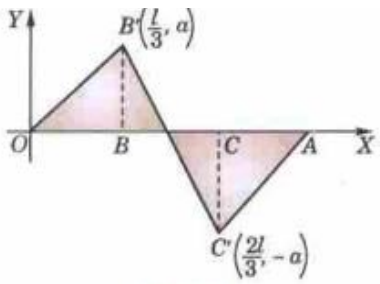


Fig. 18.2

The displacement $y(x, t)$ of any point of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(i)$$

and the boundary conditions are

$$y(0, t) = 0 \qquad ...(ii)$$

$$y(l, t) = 0 \qquad ...(iii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad ...(iv)$$

The remaining condition is that at $t = 0$, the string rests in the form of the broken line $OB'C'A$. The equation of OB' is $y = (3a/l)\,x$;

the equation of $B'C'$ is $\quad y - a = \dfrac{-2a}{(l/3)}\left(x - \dfrac{l}{3}\right)$, *i.e.*, $y = \dfrac{3a}{l}(l - 2x)$

and the equation of $C'A$ is $\quad y = \dfrac{3a}{l}(x - l)$

Hence the fourth boundary condition is

$$\left.\begin{aligned} y(x, 0) &= \frac{3a}{l}\,x,\ 0 \le x \le \frac{l}{3} \\ &= \frac{3a}{l}(l - 2x),\ \frac{l}{3} \le x \le \frac{2l}{3} \\ &= \frac{3a}{l}(x - l),\ \frac{2l}{3} \le x \le l \end{aligned}\right\} \qquad ...(v)$$

As in example 18.6, the solution of (*i*) satisfying the boundary conditions (*ii*), (*iii*) and (*iv*), is

$$y(x, t) = b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad [\text{Where } b_n = C_2C_3]$$

Adding all such solutions, the most general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad ...(vi)$$

Putting $t = 0$, we have $y(x, 0) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ $\qquad ...(vii)$

$n=1$

In order that the condition (v) may be satisfied, (v) and (vii) must be same. This requires the expansion of $y(x, 0)$ into a Fourier half-range sine series in the interval $(0, l)$.

$\therefore$ by (1) of § 10.7,

$$b_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{2l/3} \frac{3a}{l}(l-2x)\sin\frac{n\pi x}{l}\,dx + \int_{2l/3}^{l} \frac{3a}{l}(x-l)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6a}{l^2}\left[\left| x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - 1\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_0^{l/3}\right.$$

$$+\left|(l-2x)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (-2)\left\{\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{l/3}^{2l/3}$$

$$\left.+\left|(x-l)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1).\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{2l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\left[\left(-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right) + \frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} - \frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{l^2}{3n\pi}\cos\frac{n\pi}{3}\right.$$

$$\left.+\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3} - \left(\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right)\right]$$

$$= \frac{6a}{l^2}\cdot\frac{3l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18a}{n^2\pi^2}\sin\frac{n\pi}{3}[1+(-1)^n] \qquad \left[\because \sin\frac{2n\pi}{3} = \sin\left(n\pi - \frac{n\pi}{3}\right) = -(-1)^n\sin\frac{n\pi}{3}\right]$$

Thus $b_n = 0$, when n is odd.

$$= \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}, \text{ when } n \text{ is even.}$$

Hence (vi) gives

$$y(x,t) = \sum_{n=2,4,\ldots}^{\infty} \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad [\text{Take } n = 2m]$$

$$= \frac{9a}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\sin\frac{2m\pi}{3}\sin\frac{2m\pi x}{l}\cos\frac{2m\pi ct}{l} \qquad \ldots(vii)$$

Putting $x = l/2$ in (vii), we find that the displacement of the mid-point of the string, *i.e.* $y(l/2, t) = 0$, because $\sin m\pi = 0$ for all integral values of m.

This shows that the mid-point of the string is always at rest.

(3) D'Alembert's solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

Let us introduce the new independent variables $u = x + ct$, $v = x - ct$ so that y becomes a function of u and v.

Then $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$

and $$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

Similarly, $$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

Substituting in (1), we get $\frac{\partial^2 y}{\partial u \partial v} = 0$ ...(2)

Integrating (2) w.r.t. v, we get $\frac{\partial y}{\partial u} = f(u)$ ...(3)

where $f(u)$ is an arbitrary function of u. Now integrating (3) w.r.t. u, we obtain

$$y = \int f(u)\,du + \psi(v)$$

where $\psi(v)$ is an arbitrary function of v. Since the integral is a function of u alone, we may denote it by $\phi(u)$. Thus

$$y = \phi(u) + \psi(v)$$

i.e. $$y(x, t) = \phi(x + ct) + \psi(x - ct) \quad ...(4)$$

This is the *general solution of the wave equation* (1).

Now to determine ϕ and ψ, suppose initially $u(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$.

Differentiating (4) w.r.t. t, we get $\frac{\partial y}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct)$

At $t = 0$, $\phi'(x) = \psi'(x)$ ...(5)

and $y(x, 0) = \phi(x) + \psi(x) = f(x)$ ...(6)

(5) gives, $\phi(x) = \psi(x) + k$

$\therefore$ (6) becomes $2\psi(x) + k = f(x)$

or $$\psi(x) = \frac{1}{2}[f(x) - k] \text{ and } \phi(x) = \frac{1}{2}[f(x) + k]$$

Hence the solution of (4) takes the form

$$y(x, t) = \frac{1}{2}[f(x + ct) + k] + \frac{1}{2}[f(x - ct) - k] = f(x + ct) + f(x - ct) \quad ...(7)$$

which is the *d'Alembert's solution* of the wave equation* (1) *(V.T.U., 2011 S)*

Example 18.9. *Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.* (V.T.U., 2011)

Solution. By d'Alembert's method, the solution is

$$y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)]$$
$$= \frac{1}{2}[k\{\sin (x + ct) - \sin 2(x + ct)\} + k\{\sin (x - ct) - \sin 2(x - ct)\}]$$
$$= k[\sin x \cos ct - \sin 2x \cos 2ct]$$

Also $\quad y(x, 0) = k(\sin x - \sin 2x) = f(x)$

and $\quad \partial y(x, 0)/\partial t = k(-c \sin x \sin ct + 2c \sin 2x \sin 2ct)_{t=0} = 0$

i.e., the given boundary conditions are satisfied.

18.5 (1) ONE-DIMENSIONAL HEAT FLOW

Consider a homogeneous bar of uniform cross-section $\alpha(\text{cm}^2)$. Suppose that the sides are covered with a material impervious to heat so that the stream lines of heat-flow are all parallel and perpendicular to the area α. Take one end of the bar as the origin and the direction of flow as the positive x-axis (Fig. 18.3). Let ρ be the density (gr/cm^3), s the specific heat (cal./gr. deg.) and k the thermal conductivity (cal./cm. deg. sec.).

Let $u(x, -t)$ be the temperature at a distance x from O. If δu be the temperature change in a slab of thickness δx of the bar, then by § 12.7 (*ii*) p. 466, the quantity of heat in this slab $= s\rho\alpha\, \delta x \delta u$. Hence the rate of increase of heat in this slab, *i.e.*, $s\rho\alpha\delta x \frac{\partial u}{\partial t} = R_1 - R_2$, where R_1 and R_2 are respectively the rate (cal./sec.) of inflow and outflow of heat.

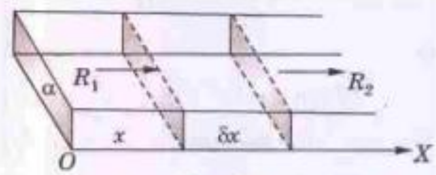


Fig. 18.3

Now by (*A*) of p. 466, $R_1 = -k\alpha\left(\frac{\partial u}{\partial x}\right)_x$ and $R_2 = -k\alpha\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$

the negative sign appearing as a result of (*i*) on p. 466.

Hence $s\rho\alpha\delta x \frac{\partial u}{\partial t} = -k\alpha\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$ *i.e.*, $\frac{\partial u}{\partial t} = \frac{k}{s\rho}\left\{\frac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x}\right\}$

Writing $k/s\rho = c^2$, called the *diffusivity* of the substance (cm^2/sec.), and taking the limit as $\delta x \to 0$, we get

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{c}^2 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} \qquad ...(1)$$

This is the *one-dimensional heat-flow equation.* (*V.T.U., 2011*)

(2) Solution of the heat equation. Assume that a solution of (1) is of the form

$$u(x, t) = X(x) \,.\, T(t)$$

where X is a function of x alone and T is a function of t only.

Substituting this in (1), we get

$$XT' = c^2X''T, \; i.e., \; X''/X = T'/c^2T \qquad ...(2)$$

Clearly the left side of (2) is a function of x only and the right side is a function of t alone. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \quad ...(3) \qquad \text{and} \qquad \frac{dT}{dt} - kc^2T = 0 \qquad ...(4)$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$X = c_1 e^{px} + c_2 e^{-px},\ T = c_3 e^{c^2 p^2 t}\ ;$$

(*ii*) *When k is negative* and $= -p^2$, say :

$$X = c_4 \cos px + c_5 \sin px,\ T = c_6 e^{-c^2 p^2 t}\ ;$$

(*iii*) *When k is zero :*

$$X = c_7 x + c_8,\ T = c_9.$$

Thus the various possible solutions of the heat-equation (1) are

$$u = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{c^2 p^2 t} \qquad ...(5)$$

$$u = (c_4 \cos px + c_5 \sin px) c_6 e^{-c^2 p^2 t} \qquad ...(6)$$

$$u = (c_7 x + c_8) c_9 \qquad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we are dealing with problems on heat conduction, it must be a transient solution, *i.e.*, u is to decrease with the increase of time t. Accordingly, the solution given by (6), *i.e.*, of the form

$$u = (C_1 \cos px + C_2 \sin px) e^{-c^2 p^2 t} \qquad ...(8)$$

is the only suitable solution of the heat equation.

Example 18.10. *Solve the equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *with boundary conditions* $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$ *and* $u(1, t) = 0$, *where* $0 < x < 1$, $t > 0$.

Solution. The solution of the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ ...(*i*)

is $\qquad u(x, t) = (c_1 \cos px + c_2 \sin px)\ e^{-p^2 t}$ ...(*ii*)

When $x = 0$, $\qquad u(0, t) = c_1 e^{-p^2 t} = 0 \quad i.e., \quad c_1 = 0.$

$\therefore$ (*ii*) becomes $\qquad u(x, t) = c_2 \sin px e^{-p^2 t}$ ...(*iii*)

When $x = 1$, $\qquad u(1, t) = c_2 \sin p \,.\, e^{-p^2 t} = 0$ or $\sin p = 0$

i.e., $\qquad p = n\pi.$

$\therefore$ (*iii*) reduces to $\qquad u(x, t) = b_n e^{-(n\pi)^2 t} \sin n\pi x$ where $b_n = c_2$

Thus the general solution of (*i*) is $u(x, t) = \sum b_n e^{-n^2 \pi^2 t} \sin n\pi x$ ...(*iv*)

When $t = 0$, $3 \sin n\pi x = u(0, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 t} \sin n\pi x$

Comparing both sides, $b_n = 3$

Hence from (*iv*), the desired solution is

$$u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \sin n\pi x.$$

Example 18.11. *Solve the differential equation* $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation, subject to the following conditions :*

(*i*) *u is not infinite for* $t \to \infty$, (*ii*) $\frac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$,

(*ii*) $u = lx - x^2$ *for* $t = 0$, *between* $x = 0$ *and* $x = l$. (P.T.U., 2007)

Solution. Substituting $u = X(x)T(t)$ in the given equation, we get

$$XT' = \alpha^2 X''T \quad i.e., \quad X''/X = \frac{T'}{\alpha^2 T} = -k^2 \text{ (say)}$$

$$\therefore \qquad \frac{d^2X}{dx^2} + k^2X = 0 \text{ and } \frac{dT}{dt} + k^2\alpha^2T = 0 \qquad ...(1)$$

Their solutions are $X = c_1 \cos kx + c_2 \sin kx,\ T = c_3 e^{-k^2\alpha^2 t}$ ...(2)

If k^2 is changed to $-k^2$, the solutions are

$$X = c_4e^{kx} + c_5e^{-kx},\ T = c_6 e^{k^2\alpha^2 t} \qquad ...(3)$$

If $k^2 = 0$, the solutions are $X = c_7x + c_8,\ T = c_9$ ...(4)

In (3), $T \to \infty$ for $t \to \infty$ therefore, u also $\to \infty$ *i.e.*, the given condition (*i*) is not satisfied. So we reject the solutions (3) while (2) and (4), satisfy this condition.

Applying the condition (*ii*) to (4), we get $c_7 = 0$.

$$\therefore \qquad u = XT = c_8c_9 = a_0 \quad \text{(say)} \qquad ...(5)$$

From (2), $\frac{\partial u}{\partial x} = (-c_1 \sin kx + c_2 \cos kx)\, kc_3e^{-k^2\alpha^2 t}$

Applying the condition (*ii*), we get $c_2 = 0$ and $-c_1 \sin kl + c_2 \cos kl = 0$

i.e., $c_2 = 0$ and $kl = n\pi$ (n an integer)

$$\therefore \qquad u = c_1 \cos kx \,.\, c_3e^{-k^2\alpha^2 t} = a_n \cos\left(\frac{n\pi x}{l}\right)\frac{e^{-n^2\pi^2\alpha^2 t}}{l^2} \qquad ...(6)$$

Thus the general solution being the sum of (5) and (6), is

$$u = a_0 + \Sigma a_n \cos(n\pi x/l)\, e^{-n^2\pi^2\alpha^2 t/l^2} \quad ...(7)$$

Now using the condition (*iii*), we get

$$lx - x^2 = a_0 + \Sigma a_n \cos(n\pi x/l)$$

This being the expansion of $lx - x^2$ as a half-range cosine series in $(0, l)$, we get

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\,dx = \frac{1}{l}\left|\frac{lx^2}{2} - \frac{x^3}{3}\right|_0^l = \frac{l^2}{6}$$

and

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\left|(lx - x^2)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)\right.$$

$$\left. - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left\{0 - \frac{l^3}{n^2\pi^2}(\cos n\pi + 1) + 0\right\} = -\frac{4l^2}{n^2\pi^2} \text{ when } n \text{ is even, otherwise } 0.$$

Hence taking $n = 2m$, the required solution is

$$u = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-4m^2\pi^2\alpha^2 t/l^2}.$$

Example 18.12. (*a*) *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.* *(U.P.T.U., 2005)*

(*b*) *Solve the above problem if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.* *(Madras, 2000 S)*

Solution. (*a*) Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad ...(i)$$

Prior to the temperature change at the end B, when $t = 0$, the heat flow was independent of time (*steady state condition*). When u depends only on x, (*i*) reduces to $\partial^2 u/\partial x^2 = 0$.

Its general solution is $u = ax + b$ ...(*ii*)

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, therefore, (*ii*) gives $b = 0$ and $a = 100/l$.

Thus the *initial condition* is expressed by $u(x, 0) = \dfrac{100}{l}x$ ...(*iii*)

Also the *boundary conditions* for the subsequent flow are

$$u(0, t) = 0 \text{ for all values of } t \quad ...(iv)$$

and

$$u(l, t) = 0 \text{ for all values of } t \quad ...(v)$$

Thus we have to find a temperature function $u(x, t)$ satisfying the differential equation (*i*) subject to the initial condition (*iii*) and the boundary conditions (*iv*) and (*v*).

Now the solution of (*i*) is of the form

$$u(x, t) = (C_1\cos px + C_2\sin px)e^{-c^2p^2t} \quad ...(vi)$$

By (*iv*), $\quad u(0, t) = C_1 e^{-c^2p^2t} = 0$, for all values of t.

Hence $C_1 = 0$ and (*vi*) reduces to $u(x, t) = C_2 \sin px \,.\, e^{-c^2p^2t}$ ...(*vii*)

Applying (*v*), (*vii*) gives $u(l, t) = C_2 \sin pl \,.\, e^{-c^2p^2t} = 0$, for all values of t.

This requires $\sin pl = 0$ *i.e.*, $pl = n\pi$ as $C_2 \neq 0$. $\quad\therefore\quad p = n\pi/l$, where n is any integer.

Hence (*vii*) reduces to $u(x, t) = b_n \sin\dfrac{n\pi x}{l}\cdot e^{-c^2n^2\pi^2t/l^2}$, where $b_n = C_2$.

[These are the solutions of (i) satisfying the boundary conditions (iv) and (v). These are the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$, of the problem.]

Adding all such solutions, the most general solution of (i) satisfying the boundary conditions (iv) and (v) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t/l^2} \qquad ...(viii)$$

Putting $t = 0$, $$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad ...(ix)$$

In order that the condition (iii) may be satisfied, (iii) and (ix) must be same. This requires the expansion of $100x/l$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \cdot \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1)\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right]_0^l = \frac{200}{l^2}\left(-\frac{l^2}{n\pi}\cos n\pi\right) = \frac{200}{n\pi}(-1)^{n+1}$$

Hence $(viii)$ gives $u(x, t) = \dfrac{200}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cdot e^{-(cn\pi/l)^2 t}$

(b) Here the initial condition remains the same as (iii) above, and the boundary conditions are

$$u(0, t) = 20 \text{ for all values of } t \qquad ...(x)$$

$$u(l, t) = 80 \text{ for all values of } t \qquad ...(xi)$$

In part (a), the boundary values (*i.e.*, the temperature at the ends) being zero, we were able to find the desired solution easily. Now the boundary values being non-zero, we have to modify the procedure.

We split up the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_s(x) + u_t(x, t) \qquad ...(xii)$$

where $u_s(x)$ is a solution of (i) involving x only and satisfying the boundary conditions (x) and (xi); $u_t(x, t)$ is then a function defined by (xii). Thus $u_s(x)$ is a steady state solution of the form (ii) and $u_t(x, t)$ may be regarded as a transient part of the solution which decreases with increase of t.

Since $u_s(0) = 20$ and $u_s(l) = 80$, therefore, using (ii) we get

$$u_s(x) = 20 + (60/l)x \qquad ...(xiii)$$

Putting $x = 0$ in (xii), we have by (x),

$$u_t(0, t) = u(0, t) - u_s(0) = 20 - 20 = 0 \qquad ...(xiv)$$

Putting $x = l$ in (xii), we have by (xi),

$$u_t(l, t) = u(l, t) - u_s(l) = 80 - 80 = 0 \qquad ...(xv)$$

Also $$u_t(x, 0) = u(x, 0) - u_s(x) = \frac{100x}{l} - \left(\frac{60x}{l} + 20\right) \qquad \text{[by } (iii) \text{ and } (xiii)]$$

$$= \frac{40x}{l} - 20 \qquad ...(xvi)$$

Hence (xiv) and (xv) give the boundary conditions and (xvi) gives the initial condition relative to the transient solution. Since the boundary values given by (xiv) and (xv) are both zero, therefore, as in part (a), we have $u_t(x, t) = (C_1 \cos px + C_2 \sin px)\, e^{-c^2p^2t}$

By (xiv), $u_t(0, t) = C_1 e^{-c^2p^2t} = 0$, for all values of t.

Hence $C_1 = 0$ and $$u_t(x, t) = C_2 \sin px \,.\, e^{-c^2p^2t} \qquad ...(xvii)$$

Applying (xv), it gives $u_t(l, t) = C_2 \sin pl e^{-c^2p^2t} = 0$ for all values of t.

This requires $\sin pl = 0$, *i.e.* $pl = n\pi$ as $C_2 \neq 0$. $p = n\pi/l$, when n is any integer.

Hence $(xvii)$ reduces to $u_t(x, t) = b_n \sin \dfrac{n\pi x}{l} e^{-c^2n^2\pi^2t/l^2}$ where $b_n = C_2$.

Adding all such solutions, the most general solution of (*xvii*) satisfying the boundary conditions (*xiv*) and (*xv*) is

$$u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-c^2n^2\pi^2t/l^2} \quad ...(xviii)$$

Putting $t = 0$, we have $u_t(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ ...(*xix*)

In order that the condition (*xvi*) may be satisfied, (*xvi*) and (*xix*) must be same. This requires the expansion of $(40/l)\,x - 20$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{40x}{l} - 20 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where} \quad b_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right)\sin\frac{n\pi x}{l}\,dx = -\frac{40}{nx}(1 + \cos n\pi)$$

i.e., $b_n = 0$, when n is odd ; $= -80/n\pi$, when n is even

Hence (*xviii*) becomes $u_t(x, t) = \sum_{n=2,4...}^{\infty} \left(\frac{-80}{n\pi}\right)\sin\frac{n\pi x}{l}\cdot e^{-c^2n^2\pi^2t/l^2}$ [Take $n = 2m$]

$$= -\frac{40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l}\cdot e^{-4c^2m^2\pi^2t/l^2} \quad ...(xx)$$

Finally combining (*xiii*) and (*xx*), the required solution is

$$u(x, t) = \frac{40x}{l} + 20 - \frac{40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l}\cdot e^{-4c^2m^2\pi^2t/l^2}.$$

Example 18.13. *The ends A and B of a rod 20 cm long have the temperature at 30°C and 80°C until steady-state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.*

Solution. Let the heat equation be $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ ...(*i*)

In steady state condition, u is independent of time and depends on x only, (*i*) reduces to

$$\partial^2 u/\partial x^2 = 0. \quad ...(ii)$$

Its solution is $u = a + bx$

Since $u = 30$ for $x = 0$ and $u = 80$ for $x = 20$, therefore $a = 30$, $b = (80 - 30)/20 = 5/2$

Thus the initial conditions are expressed by

$$u(x, 0) = 30 + \frac{5}{2}x \quad ...(iii)$$

The boundary conditions are $u(0, t) = 40$, $u(20, t) = 60$

Using (*ii*), the steady state temperature is

$$u(x, 0) = 40 + \frac{60-40}{20}x = 40 + x \quad ...(iv)$$

To find the temperature u in the intermediate period,

$$u(x, t) = u_s(x) + u_t(x, t)$$

where $u_s(x)$ is the steady state temperature distribution of the form (*iv*) and $u_t(x, t)$ is the transient temperature distribution which decreases to zero as t increases.

Since $u_t(x, t)$ satisfies one dimensional heat equation

$$\therefore \quad u(x, t) = 40 + x + \sum_{n=1}^{\infty}(a_n \cos px + b_n \sin px)\,e^{-p^2t} \quad ...(v)$$

$$u(0, t) = 40 = 40 + \sum_{n=1}^{\infty} a_n e^{-p^2t} \quad \text{whence } a_n = 0.$$

$\therefore$ (v) reduces to $$u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin pxe^{-p^2 t} \quad ...(vi)$$

Also $$u(20, t) = 60 = 40 + 20 + \sum_{n=1}^{\infty} b_n \sin 20\, pe^{-p^2 t}$$

or $$\sum_{n=1}^{\infty} b_n \sin 20\, pe^{-p^2 t} = 0 \text{ i.e., } \sin 20p = 0 \text{ i.e., } p = n\pi/20$$

Thus (vi) becomes $$u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-n\pi t/20} \quad ...(vii)$$

Using (iii), $$30 + \frac{5}{2}x = u(0, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

or $$\frac{3x}{2} - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

where $b_n = \frac{2}{20} \int_0^{20} \left(\frac{3x}{2} - 10\right) \sin \frac{n\pi x}{20}\, dx = -\frac{20}{n\pi}(1 + 2\cos n\pi)$

Hence from (vii), the desired solution is

$$u = 40 + x - \frac{20}{\pi} \sum \frac{1 + 2\cos n\pi}{n} \sin \frac{n\pi x}{20} e^{-(n\pi/20)^2 t}.$$

Example 18.14. Bar with insulated ends. *A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.*

Solution. The temperature $u(x, t)$ along the bar satisfies the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ...(i)$$

By law of heat conduction, the rate of heat flow is proportional to the gradient of the temperature. Thus, if the ends $x = 0$ and $x = l$ (= 100 cm) of the bar are insulated (Fig. 18.4) so that no heat can flow through the ends, the boundary conditions are

$$\frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(l, t)}{\partial x} = 0 \text{ for all } t \quad ...(ii)$$

Initially, under steady state conditions, $\frac{\partial^2 u}{\partial x^2} = 0$. Its solution is $u = ax + b$.

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$ $\therefore$ $b = 0$ and $a = 1$.

Thus the initial condition is $u(x, 0) = x \quad 0 < x < l$. ...(*iii*)

Now the solution of (*i*) is of the form $u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-c^2 p^2 t}$ ...(*iv*)

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px) e^{-c^2 p^2 t} \quad \text{...}(v)$$

Putting $x = 0$, $\left(\frac{\partial u}{\partial x}\right)_0 = c_2 p e^{-c^2 p^2 t} = 0$ for all t. [By (*ii*)]

$\therefore$ $c_2 = 0$

Putting $x = l$ in (*v*), $\left(\frac{\partial u}{\partial x}\right)_l = -c_1 p \sin pl e^{-c^2 p^2 t}$ for all t. [By (*ii*)]

$\therefore$ $c_1 p \sin pl = 0$ *i.e.*, p being $\neq 0$, either $c_1 = 0$ or $\sin pl = 0$.

When $c_1 = 0$, (*iv*) gives $u(x, t) = 0$ which is a trivial solution, therefore $\sin pl = 0$.

or $pl = n\pi$ or $p = n\pi/l$, $n = 0, 1, 2, \ldots$.

Hence (*iv*) becomes $u(x, t) = c_1 \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$.

$\therefore$ the most general solution of (*i*) satisfying the boundary conditions (*ii*) is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} \quad (\text{where } A_n = c_1) \quad \text{...}(vi)$$

Putting $t = 0$, $u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} = x$ [by (*iii*)]

This requires the expansion of x into a half range cosine series in $(0, l)$.

Thus $$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x/l \qquad \text{where } a_0 = \frac{2}{l}\int_0^l x\,dx = l$$

and $$a_n = \frac{2}{l}\int_0^l x \cos \frac{n\pi x}{l}\,dx = \frac{2l}{n^2\pi^2}(\cos n\pi - 1)$$

$= 0$, where n is even ; $= -4l/n^2\pi^2$, when n is odd.

$\therefore$ $A_0 = \frac{a_0}{2} = l/2$, and $A_n = a_n = 0$ for n even ; $= -4l/n^2\pi^2$ for n odd.

Hence (*vi*) takes the form

$$u(x, t) = \frac{l}{2} + \sum_{n=1, 3, \ldots}^{\infty} \frac{4l}{n^2\pi^2} \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$$

$$= \frac{l}{2} - \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{-c^2 (2n-1)^2 \pi^2 t/l^2} \quad \text{...}(vii)$$

This is the required temperature at a point P_1 distant x from end A at any time t.

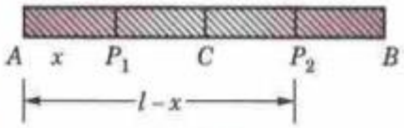


Fig. 18.4

Obs. *The sum of the temperatures at any two points equidistant from the centre is always 100°C, a constant.*

Let P_1, P_2 be two points equidistant from the centre C of the bar so that $CP_1 = CP_2$ (Fig. 18.4).

If $\quad AP_1 = BP_2 = x \quad$ (say), then $AP_2 = l - x$.

$\therefore$ Replacing x by $l - x$ in (*vii*), we get the temperature at P_2 as

$$u(l-x, t) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos\frac{(2n-1)\pi(l-x)}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}}$$

$$= \frac{l}{2} + \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos\frac{(2n-1)\pi x}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}} \qquad ...(viii)$$

$$\left\{\because \quad \cos\frac{(2n-1)\pi(l-x)}{l} = \cos\left[2n\pi - \pi - \frac{(2n-1)\pi x}{l}\right] = -\cos\frac{(2n-1)\pi x}{l}\right.$$

Adding (*vii*) and (*viii*), we get $u(x, t) + u(l-x, t) = l = 100°C$.

Module 3

Complex Variable - Differentiation

Complex number

Complex number is of the form

$Z = x+iy$ where x is real part, y-imaginary part.

$x = \mathrm{Re}\, z \qquad y = \mathrm{Im}\, z$

Two complex numbers are equal if and only if their real parts are equal and their imaginary part are equal.

$i = (0,1)$

If $Z = x+iy \qquad \bar{Z} = x-iy$

$|z| = \sqrt{x^2+y^2} \qquad z^2 = -1 \Rightarrow z = \pm\sqrt{-1} = \pm i.$

unit circle

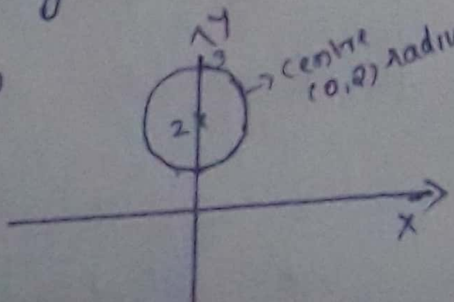
unit circle can be represented by

$|z| = 1$

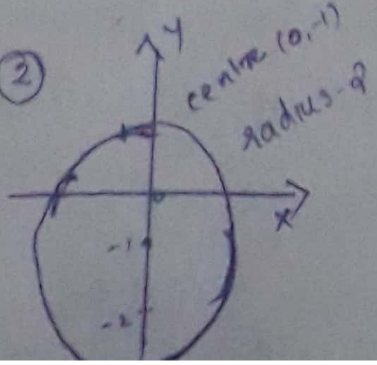
$|z-a| = \rho$ Circle with centre a and radius ρ.

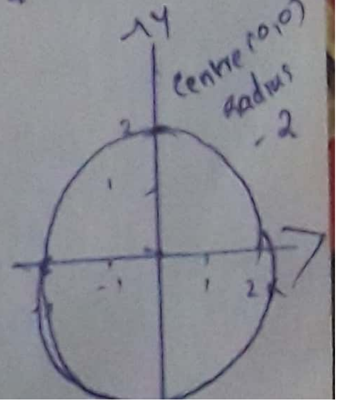
Q1) Find the region in z plane represented by (1) $|z-2i| = 1$ (2) $|z+i| = 2$ (3) $|z| = 2$

→ (1)



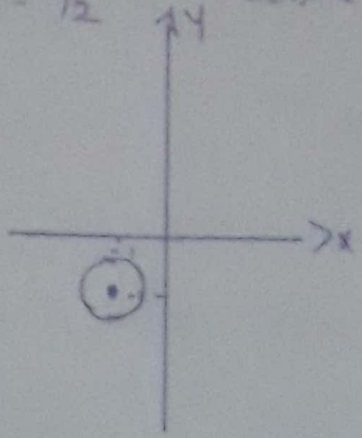
(2)





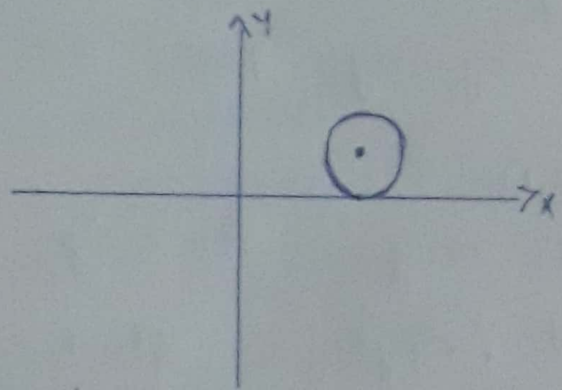
(2) $|z+1+i| = 1/2$

→ $|z-(-1-i)| = 1/2$ Centre $-1-i$ $(-1,-1)$ Radius $= 1/2$



(3) $|z-(2+i)| = 1$

Centre = $(2+i)$ = $(2,1)$ radius → 1



Complex functions

S is a set of complex numbers and a function f is defined on S is a rule that assigns to every z in S a complex number w, called the value of f at z. We write

$w = f(z)$ Here z varies in S and is called a complex variable. The set 'S' is called the domain of f. The set of all values of a function f is called the range of f.

W is complex then we write

$W = u+iv$ u is the real part
v is the imaginary part

$w = f(z) = u(x,y) + i\,v(x,y)$

Q(a) $w = f(z) = z^2 + 3z$ find u and v and calculate the value of f at $z = 1+3i$

Ans:

$f(z) = (x+iy)^2 + 3(x+iy)$

$= x^2 + i\,2xy + i^2y^2 + 3x + i\,3y$

$= x^2 + i\,2xy - y^2 + 3x + i\,3y$

$= [x^2 - y^2 + 3x] + i[2xy + 3y]$

$\quad u \qquad + i\,v$

$u = x^2 - y^2 + 3x$ & $v = 2xy + 3y$

$f(1+3i) = (1+3i)^2 + 3(1+3i)$

$= 1 + 6i - 9 + 3 + 9i = \underline{\underline{-5 + 15i}}$

2) $w = f(z) = 2iz + 6\bar{z}$ find u and v and the value of f at $z = \frac{1}{2} + 4i$.

$f(z) = 2i(x+iy) + 6(x - iy)$

$= 2ix - 2y + 6x - i6y$

$= (6x - 2y) + i(2x - 6y)$

$u = \underline{\underline{6x - 2y}}$ $\quad v = \underline{\underline{2x - 6y}}$

$f(\frac{1}{2} + 4i) = 2i(\frac{1}{2} + 4i) + 6(\frac{1}{2} - 4i)$

$= i - 8 + 3 - 24i$

$= \underline{\underline{-5 - 23i}}$.

Q.w $f(z) = 5z^2 - 12z + 3 + 2i$

Find u and v and calculate the value of f at $z = 4 - 3i$.

3) If $f(z) = \frac{1}{1+z}$ find real part of f and imaginary part of f and their value at $z = 1-i$

$w = f(z)$

$= \frac{1}{1+z}$

$u+iv = \frac{1}{1+(x+iy)} = \frac{1}{(1+x)+iy} = \frac{(1+x)-iy}{[(1+x)+iy][(1+x)-iy]}$

$= \frac{1+x - iy}{(1+x)^2+y^2}$

$= \frac{1+x}{(1+x)^2+y^2} - i\frac{y}{(1+x)^2+y^2}$

$u(x,y) = \frac{1+x}{(1+x)^2+y^2}$ $\quad v(x,y) = \frac{-y}{(1+x)^2+y^2}$

$z = 1-i \quad x=1 \quad y=-1$

$u = \frac{1+1}{(1+1)^2+1^2} = \frac{2}{5}$ $\quad v = \frac{-1}{(1+1)^2+1^2} = \frac{1}{5}$

$\therefore f = u+iv = \frac{2}{5}+\frac{1}{5}i$

H.W $f(z) = \frac{z-1}{z+1}$ at $z = 2i$.

Limit, continuity

A function $f(z)$ is said to have the limit l as z approaches a point z_0, written

$\lim_{z\to z_0} f(z) = l$

if f is defined in a neighbourhood of z_0 and if the values of f are close to l for all z close to z_0

A function $f(z)$ is said to be continuous if (i) $\lim f(z) = l$ exist and $\lim_{z\to z_0} f(z) = f(z_0)$

Derivative

The derivative of a complex function f at a point z_0 is written $f'(z_0)$ and is defined by $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

Provided this limit exist. Then f is said to be differentiable at z_0.

Or $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ $\left\{ \begin{array}{l} \text{Take} \\ \Delta z = z - z_0 \end{array} \right.$

Pbms

1 S.T $\lim_{z \to 0} \frac{z}{\bar{z}}$ doesnot exist.

→ $\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{(x,y) \to 0} \frac{x+iy}{x-iy}$

Take $y = mx$.

$$\lim_{x \to 0} \frac{x+imx}{x-imx} = \lim_{x \to 0} \frac{x(1+im)}{x(1-im)} = \frac{1+im}{1-im}$$

It depends on m. ∴ limit doesnot exist.

2 Check whether $\lim_{z \to 0} \left(\frac{z}{\bar{z}}\right)^2$ exist or not

Ans $\lim_{z \to 0} \left(\frac{z}{\bar{z}}\right)^2 = \lim_{(x,y) \to 0} \left(\frac{x+iy}{x-iy}\right)^2$

$y = mx$. $\lim_{x \to 0} \left[\frac{x+imx}{x-imx}\right]^2 = \lim_{x \to 0} \left[\frac{x(1+im)}{x(1-im)}\right]^2$

$= \frac{(1+im)^2}{(1-im)^2}$ depends on $m \Rightarrow$ limit doesnot exist

(3) Check whether the following functions are continuous or not at $z=0$

1) $f(z) = \begin{cases} \dfrac{Re(z)}{|z|} & , z \neq 0 \\ 0 & z=0 \end{cases}$

→ Continuity → (1) limit exist

(2) $\lim\limits_{z \to z_0} f(z) = f(z_0)$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{Re(z)}{|z|} = \lim_{z \to 0} \frac{x}{\sqrt{x^2+y^2}}$$

$y = mx$ $\quad \lim\limits_{x \to 0} \dfrac{x}{\sqrt{x^2+(mx)^2}} = \lim\limits_{x \to 0} \dfrac{x}{x\sqrt{1+m^2}}$

depends on m ⇒ limit doesnot exist
⇒ function discontinuous.

(4). $f(z) = \begin{cases} \dfrac{Re(z^2)}{|z|} & z \neq 0 \\ 0 & z=0 \end{cases}$

Ans: $\lim\limits_{z \to 0} f(z) = \lim\limits_{z \to 0} \dfrac{Re(z^2)}{|z|}$ $\qquad z^2 = (x+iy)^2 = x^2 y^2 + i2xy$

$$= \lim_{(x,y) \to 0} \frac{x^2 y^2}{\sqrt{x^2+y^2}}$$

$y = mx$. $\quad \lim\limits_{x \to 0} \dfrac{x^2 - m^2 x^2}{\sqrt{x^2+m^2x^2}} = \lim\limits_{x \to 0} \dfrac{x^2(1-m^2)}{x\sqrt{1+m^2}}$

$= 0$ doesnot depends on m

⇒ limit exist

$f(z_0) = f(0) = 0$.

limit exist and
$\lim\limits_{z \to 0} f(z) = f(z_0)$ ⇒ function $f(z)$ is
continuous at $z=0$

(3) $f(z) = \begin{cases} \dfrac{\operatorname{Im}(z^2)}{|z|^2} & , z \neq 0 \\ 0 & z = 0 \end{cases}$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\operatorname{Im}(z^2)}{|z|^2} = \lim_{(x,y) \to 0} \frac{2xy}{x^2+y^2}$$

$y = mx \longrightarrow \lim_{x \to 0} \dfrac{2x \times mx}{x^2 + m^2x^2} = \dfrac{2mx^2}{x^2(1+m^2)}$

= depends on $m \Rightarrow$ limit doesnot exist $\Rightarrow f(z)$ not continuous at $z=0$

(4) $f(z) = \begin{cases} |z|^2 \operatorname{Im}\left(\frac{1}{z}\right) & z \neq 0 \\ 0 & z = 0. \end{cases}$

5) $f(z) = \begin{cases} \dfrac{\operatorname{Im}(z)}{|z|} & z \neq 0. \\ 0 & z = 0. \end{cases}$

1) S.T $f(z) = z^2$ is differentiable for all z

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\frac{2z\Delta z + \Delta z^2}{\Delta z} \right) = \lim_{\Delta z \to 0} \frac{\Delta z\,[2z + \Delta z]}{\Delta z}$$

$$= \underline{\underline{2z}}$$

$\therefore f(z) = z^2$ is differentiable everywhere.

2) Check the differentiability of $f(z) = |z|^2$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} \qquad |z|^2 = z\bar{z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\overline{z+\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z} \qquad \overline{z+\Delta z} = \bar{z}+\overline{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z\bar{z}} + z\overline{\Delta z} + \Delta z\,\bar{z} + \Delta z \cdot \overline{\Delta z} - \cancel{z\bar{z}}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}$$

The limit doesnot exist $\Rightarrow$ $f(z)$ is not differentiable.

(3) Check the differentiablity of $\bar{z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(\overline{z+\Delta z}) - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(\cancel{\bar{z}} + \overline{\Delta z} - \cancel{\bar{z}})}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

limit doesnot exist

$\therefore$ The function is discontinuous

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Analytic function

A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D.

Cauchy - Reimann Equations

Let $w = f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if its partial derivatives exists and satisfy the conditions

$$U_x = V_y \quad \& \quad U_y = -V_x$$

Pbms

1 Show that $f(z) = z^2$ is analytic for all z.

Ans $f(z) = z^2$ is analytic if C-R eqns are satisfied

$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$$

$U = x^2 - y^2$ $\qquad V = 2xy$

$U_x = 2x$ $\qquad V_x = 2y$

$U_y = -2y$ $\qquad V_y = 2x$

$\therefore U_x = V_y$ & $U_y = -V_x$

C.R equations satisfied

$\Rightarrow f(z)$ is analytic

2 Show that $f(z) = e^z$ is analytic everywhere.

Ans $f(z) = e^z$

$= e^{x+iy} = e^x e^{iy} = e^x[\cos y + i\sin y]$

$u = e^x \cos y \qquad v = e^x \sin y$

$u_x = e^x \cos y \qquad v_x = e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = e^x \cos y$

$u_x = v_y$ & $u_y = -v_x$ $\Rightarrow$ C.R eqns Satisfied

$f(z)$ is analytic

3 Test the analyticity of $f(z) = Re(z^2) - Im(z^2)$

$z^2 = (x+iy)^2$
$= x^2 - y^2 + i2xy$

Ans. $f(z) = x^2 - y^2 - 2xy$

$u = x^2 - y^2 - 2xy \qquad v = 0$

$u_x = 2x - 2y \qquad v_x = 0$

$\qquad v_y = 0$

$u_y = -2y - 2x$

C-R eqns are not Satisfied

$\Rightarrow f(z)$ not analytic

4 $w = \sin z$

$u + iv = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + \cos x \, i \sinh y$ $\quad \left\{ \begin{array}{l} \sin(ix) = i\sinh x \\ \cos(ix) = \cosh x \end{array} \right.$

$= \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y \qquad v = \cos x \sinh y$

$\qquad v_x = -\sin x \sinh y$

$u_x = \cos x \cosh y \qquad v_y = \cos x \cosh y$

$u_y = +\sin x \sinh y$

$u_x = v_y$ & $u_y = -v_x$ $\Rightarrow$ C.R eqns Satisfied

$f(z) = \sin z$ is analytic

5) $w = \cosh z$

$u + iv = \cos(iz)$

$= \cos i(x+iy) = \cos(ix - y)$

$= \cos ix \cos y + \sin(ix) \sin y$

$= \cosh x \cos y + i \sinh x \sin y$

$u = \cosh x \cos y$ & $v = \sinh x \sin y$

$u_x = \sinh x \cos y$ $\quad v_x = \cosh x \sin y$

$u_y = -\cosh x \sin y$ $\quad v_y = \sinh x \cos y$

C-R equations satisfied

$u_x = v_y$ & $u_y = -v_x$

$\Rightarrow f(z)$ is analytic

H.W. 1) $f(z) = z\bar{z}$

(2) $f(z) = i z\bar{z}$

(3) $w = \cos z$

(4) $w = \sinh z$

Laplace equation

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D then u and v satisfy the Laplace equation.

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \text{and} \quad \nabla^2 v = v_{xx} + v_{yy} = 0$$

Note

1) Solutions of Laplace equation having continuous 2nd order partial derivatives are called harmonic functions.

2) The real and imaginary parts of analytic functions are harmonic functions

8)

Pblms

1 Verify that $u(x,y) = x^2 - y^2$ is harmonic and find its harmonic conjugate. Also find the associated analytic function.

Ans

$u(x,y) = x^2 - y^2$

$u_x = 2x \qquad u_{xx} = 2$

$u_y = -2y \qquad u_{yy} = -2$

$\nabla^2 u = u_{xx} + u_{yy} = 2 - 2 = 0$

Laplace equation satisfied

$\therefore$ u is a harmonic function.

C-R equations $u_x = v_y$ & $u_y = -v_x$

$\boxed{v_y = 2x} \qquad \boxed{v_x = 2y}$

$v_x = 2y$

integrating both side w.r.to x.

$\int v_x \, dx = \int 2y \, dx$

$v = 2xy + h(y) \longrightarrow (1)$

diff (1) w.r.to y

$v_y = 2x + h'(y)$

We know that $v_y = 2x$

$\therefore 2x + h'(y) = 2x \Rightarrow h'(y) = 0$

$\Rightarrow h(y) = c$

(1) $\Rightarrow v = \underline{2xy + c}$

$f(z) = u+iv = x^2-y^2+i(2xy+c)$

2 Verify that $u = e^x \cos y$ is harmonic and find its harmonic conjugate. Also find $f(z)$

Ans:

$u = e^x \cos y$

$u_x = e^x \cos y \qquad u_y = -e^x \sin y$

$u_{xx} = e^x \cos y \qquad u_{yy} = -e^x \cos y$

$u_{xx} + u_{yy} = 0 \implies u$ harmonic

By C.R eqns

$u_x = v_y$ & $u_y = -v_x$

$v_y = e^x \cos y \qquad v_x = -(-e^x \sin y) = e^x \sin y \; \} \to (1)$

$v_y = e^x \cos y$

integrating w.r.to y $\qquad \boxed{v = e^x \sin y + f(x)}$

diff w.r.to x $\qquad v_x = e^x \sin y + f'(x)$

(1) $\implies e^x \sin y = e^x \sin y + f'(x) \implies f'(x) = 0$

$f(x) = c$

$$v = e^x \sin y + c$$

$$f(z) = e^x \cos y + i(e^x \sin y + c)$$

3 P.T $u = \cos x \cosh y$ is harmonic. Also find harmonic conjugate

Ans:

$u = \cos x \cosh y$

$u_x = -\sin x \cosh y \qquad u_y = \cos x \sinh y$

$u_{xx} = -\cos x \cosh y \qquad u_{yy} = \cos x \cosh y$

$u_{xx} + u_{yy} = 0 \implies u$ harmonic

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using C.R eqns

$V_y = -\sin x \cosh y$ $\quad\quad$ $V_x = -\cos x \sinh y \rightarrow (1)$

Integrating w.r.to y

$V = -\sin x \sinh y + f(x)$

diff w.r.to x

$V_x = -\cos x \sinh y + f'(x)$

$(1) \Rightarrow \quad f'(x) = 0 \Rightarrow f(x) = c$

$V = -\sin x \sinh y + c$

$f(z) = \cos x \cosh y + i(-\sin x \sinh y + c)$

4) P.T $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. Also find harmonic conjugate

Ans $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 3y^2 + 6x$ $\quad\quad$ $u_y = -6xy - 6y$

$u_{xx} = 6x + 6$ $\quad\quad$ $u_{yy} = -6x - 6$

$u_{xx} + u_{yy} = 0 \Rightarrow u$ harmonic

By C.R eqns

$V_y = u_x = \cancel{x^2 - 3xy}$ $\quad\quad$ $V_x = -u_y = 6xy + 6y \} \rightarrow (1)$

$3x^2 - 3y^2 + 6x$

$V_y = 3x^2 - 3y^2 + 6x$

Integrating w.r.to y

$V = 3x^2y - y^3 + 6xy + h(x)$

diff w.r.to x

$V_x = 6xy + 6y + h'(x)$

$(1) \Rightarrow h'(x) = 0$ $\quad\quad$ $h(x) = c.$

$V = 3x^2y - y^3 + 6xy + c$

5 S.T $u = x^3 - 3xy^2$ is harmonic. Also find the corresponding analytic function (8)

Ans

$u = x^3 - 3xy^2$

$u_x = 3x^2 - 3y^2$ $\quad u_y = -6xy$

$u_{xx} = 6x$ $\quad u_{yy} = -6x$

$u_{xx} + u_{yy} = 0 \Rightarrow u$ harmonic

using C.R eqns

$V_y = 3x^2 - 3y^2$ $\quad V_x = 6xy \longrightarrow (1)$.

$V_y = 3x^2 - 3y^2$

integrating w.r.to y

$V = 3x^2y - y^3 + f(x)$

Diff w.r.to x

$V_x = 6xy \cancel{+ 3y^2} + f'(x)$

(1) $\Rightarrow f'(x) = 0$ $\quad \Rightarrow f(x) = \cancel{6xy}\, c$

$\therefore V = \underline{\underline{3x^2y - y^3 + c}}$

$\underline{\underline{f(z) = x^3 - 3xy^2 + i(3x^2y - y^3 + c)}}$

6 If $f(z) = u(x,y) + iv(x,y)$ be an analytic function then prove the following.

1) $u(x,y) =$ a constant $\Rightarrow f(z)$ is constant

2) $V(x,y) =$ a constant $\Rightarrow f(z)$ is constant

3) $|f(z)| =$ a constant $\Rightarrow f(z)$ constant

4) $\operatorname{Arg}(f(z)) =$ a constant $\Rightarrow f(z)$ is constant

Proof

(1) $u(x,y) =$ a constant $= k$

$$\Rightarrow \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 0$$

Using C-R equations $u_x = v_y = 0$ $u_y = -v_x = 0$

$v_y = 0$ & $v_x = 0$

$\Rightarrow v(x,y) =$ a constant

u, v constant $\Rightarrow f(z) = u + iv$ Constant

(2) $v(x,y) =$ a constant $= k$

$\Rightarrow v_x = 0 \quad v_y = 0$

By using C-R eqns $u_x = 0 \quad u_y = 0 \Rightarrow u$ constant

u & v are constant $\Rightarrow f(z)$ constant

(3). $|f(z)| =$ constant $\Rightarrow |f(z)| = k \Rightarrow \sqrt{u^2+v^2} = k$

$$\Rightarrow u^2 + v^2 = k^2 \neq 0 \longrightarrow (1)$$

Diff w.r.to to x

$$2uu_x + 2vv_x = 0 \Rightarrow uu_x + vv_x = 0 \longrightarrow (2)$$

Diff (1) w.r.to y

$$\Rightarrow 2uu_y + 2vv_y = 0 \Rightarrow uu_y + vv_y = 0 \longrightarrow (3)$$

Using C-R eqns in (2) & (3) $\{v_x = -u_y \text{ \& } v_y = u_x\}$

(2) $\Rightarrow uu_x - vu_y = 0 \longrightarrow (4)$

(3) $\Rightarrow uu_y + vu_x = 0 \longrightarrow (5)$

(4) $\times u$ + (5) $\times v$ $\quad u^2u_x - uvu_y + uvu_y + v^2u_x = 0$

$$(u^2+v^2)u_x = 0 \Rightarrow u_x = 0$$

$\Rightarrow u$ independent of x

③ × u − ④ × v

$$u^2 u_y + uv u_x - \{uv u_x - v^2 u_y\} = 0$$

$$(u^2 + v^2)\, u_y = 0 \Rightarrow u_y = 0$$

$\Rightarrow u$ independent of y

$\Rightarrow u$ constant

$\Rightarrow f(z)$ constant

4) $\operatorname{Arg}(f(z)) = \tan^{-1}(v/u) = k$

$$\frac{v}{u} = \tan k \Rightarrow u = v\cot k$$

$$\Rightarrow u = v k_1 \quad \{\cot k = k_1\}$$

$$\Rightarrow u - v k_1 = 0$$

$u - vk_1$ is the real part of $(1 + ik_1)(u + iv)$

$$(1 + ik_1) f(z) = \text{constant} = k$$

$\Rightarrow f(z)$ is constant

Conformal mapping

A complex function $w = f(z)$ is called conformal if it preserves angles between oriented curves in magnitude as well as in sense of rotation.

1. Discuss the conformal mapping of $w = z^2$

Ans: Given $f(z) = z^2$

$$u + iv = (x + iy)^2 = x^2 - y^2 + i\,2xy$$

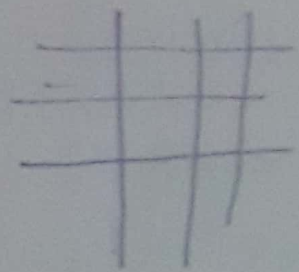
$$u = x^2 - y^2 \qquad v = 2xy$$

$\underline{x = c}$ $\Rightarrow u = c^2 - y^2$ $\quad v = 2cy$

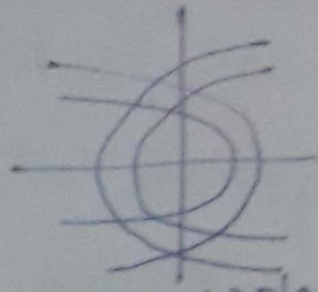
$\Rightarrow y^2 = c^2 - u$ $\quad v^2 = 4c^2y^2$

$= 4c^2(c^2 - u)$

Parabola open to the left



Z plane



w plane.

$\underline{y = k}$ $\quad u = x^2 - k^2$ $\quad v = 2xk.$

$x^2 = u + k^2$ $\quad v^2 = 4x^2k^2$

$= 4(u + k^2)k^2$

$= 4k^2(u + k^2)$

Parabola open to the right

2 Discuss the conformal mapping of $w = e^z$

Ans $W = e^z$

$u + iv = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

$u = e^x \cos y$ $\quad V = e^x \sin y$

Case I $\quad x = k$

$u = e^k \cos y$ $\quad V = e^k \sin y$

$u^2 + v^2 = e^{2k}.$ $\quad u^2 + v^2 = (e^k)^2.$

$x =$ constants maps to a circle with centre origin and radius e^k.

Case 2 $y = k$

$$u = e^x \cos k \qquad v = e^x \sin k$$

$$\frac{v}{u} = \tan k.$$

$$\tan^{-1}(v/u) = k$$

$\arg w = k$ maps to a ray

Pbms

1 Find the image of the triangle bounded by $x=1$ $y=1$ and $x+y=1$ under $w=z^2$

Ans:

$$W = z^2$$

$$u+iv = (x+iy)^2 = x^2 - y^2 + i\,2xy$$

$$u = x^2 - y^2 \qquad v = 2xy$$

$$x=1 \quad u = 1-y^2 \qquad v = 2y$$

$$y^2 = 1-u \qquad v^2 = 4y^2$$

$$= 4(1-u)$$

Image of $x=1$ is $v^2 = 4(1-u)$ parabola

$$y=1 \qquad u = x^2 - 1 \qquad v = 2x.$$

$$x^2 = u+1 \qquad v^2 = 4x^2$$

$$= 4(u+1)$$

$y=1$ to a parabola $v^2 = 4(u+1)$

$$x+y=1 \qquad u = x^2 - y^2 = (x+y)(x-y)$$

$$u = x-y \qquad v = 2xy$$

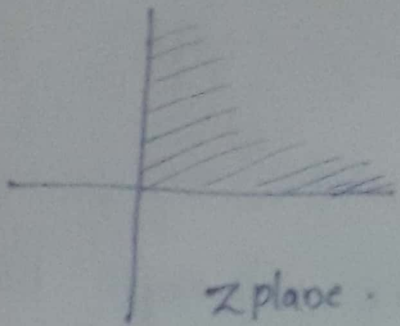
$$(x+y)^2 = (x-y)^2 + 4xy$$

$$1 = u^2 + 2v \Rightarrow \text{Parabola}$$

2 Find the image of the first quadrant of z plane under the transformation $w=z^2$

Ans) $w=z^2$

$u=x^2-y^2$ $\quad v=2xy$



z plane.

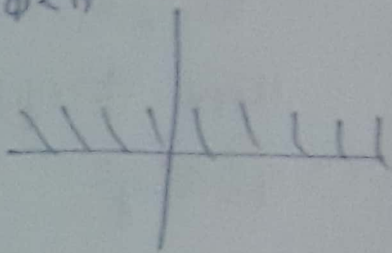
$0\le\theta\le\pi/2$.

$z=re^{i\theta}$
$w=Re^{i\phi}$
$w=z^2$
$Re^{i\phi}=r^2e^{i2\theta}$
$R=r^2$ $\quad \phi=2\theta$

$\theta=0$ $\quad \phi=0$

$\theta=\pi/2$ $\quad \phi=\pi$

$0<\phi<\pi$



wplane

3 Find the image of the strip $\frac{1}{2}\le x\le 1$ under the mapping $w=z^2$

$w=z^2 \Rightarrow u=x^2-y^2$ $\quad v=2xy$

$x=\frac{1}{2}$. $\quad v^2=4c^2(c^2-u)$

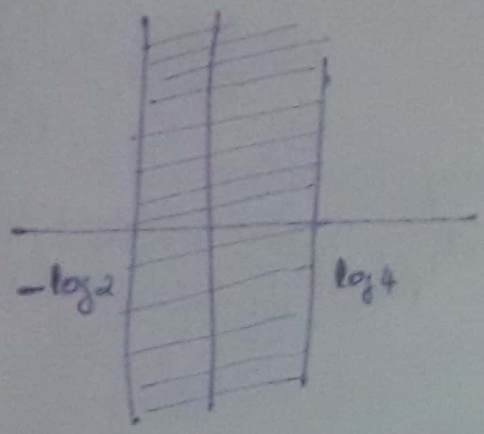
$=4\times(1/2)^2\left((\frac{1}{2})^2-u\right) = \frac{1}{4}-u$.

$v^2=\frac{1}{4}-u$ A parabola.

$x=1$ $\quad v^2=4(1-u)$ is a parabola

∴ The infinite strip $\frac{1}{2}\le x\le 1$ is mapped on to the region bounded by the parabolas $v^2=\frac{1}{4}-u$ and $v^2=4(1-u)$.

1 Find the image of the region $-\log 2 \le x \le \log 4$ under the mapping $w = e^z$.



Z plane.

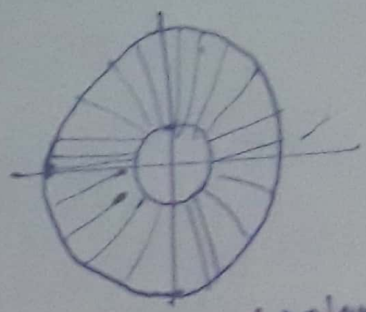
$\left\{ x = \text{constant} \text{ to } |w| = e^{x_0}. \right.$

$y = \text{constant}$ to $\arg w = y_0$

Image of the line $x = -\log 2$ is the circle

$|w| = e^{-\log 2} = e^{\log 2^{-1}} = 1/2.$

$x = \log 4$ to $|w| = e^{\log 4} = 4$



w plane.

2 Find the image of the region $-1 \le x \le 2$, $-\pi \le y \le \pi$ under $w = e^z$

Ans $x = -1$ to the circle $|w| = e^{-1}$

$x = 2$ to the circle $|w| = e^2$.

$y = -\pi$ and $y = \pi$ are mapped on the rays

$\arg w = -\pi$ and $\arg w = \pi$.

Discuss $w = \frac{1}{z}$

$z = re^{i\theta}$ $w = Re^{i\phi}$

$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$

$R = \frac{1}{r}$ $\phi = -\theta$

$|z| = 1 \longrightarrow r = 1$ $R = 1 \longrightarrow |w| = 1$ unit circle.

$u + iv = \frac{1}{x+iy} = \frac{x - iy}{x^2 + y^2}$

$u = \frac{x}{x^2+y^2}$ $v = \frac{-y}{x^2+y^2}$.

$z = \frac{1}{w}$

$x + iy = \frac{1}{u+iv}$

$x = \frac{u}{u^2+v^2}$ $y = \frac{-v}{u^2+v^2}$

Fixed points

Fixed points of a mapping $w = f(z)$ are points that are mapped on to themselves are kept fixed under the mapping.

Thus $w = f(z) = z$.

Pbms

1 $\frac{1}{4} < y < \frac{1}{2}$ under $w = \frac{1}{z}$

Ans $w = \frac{1}{z}$ $z = \frac{1}{w}$

$u + iv = \frac{1}{x+iy}$ $u = \frac{x}{x^2+y^2}$ $v = \frac{-y}{x^2+y^2}$

$x = \frac{u}{u^2+v^2}$ $y = \frac{-v}{u^2+v^2}$

$\frac{1}{4} < y \Rightarrow \frac{1}{4} < \frac{-v}{u^2+v^2} \Rightarrow u^2+v^2 < -4v$

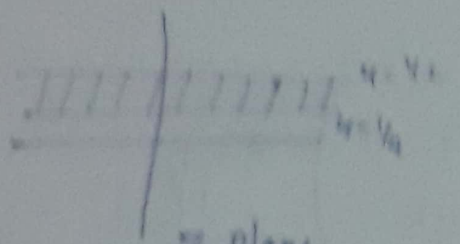
$u^2+v^2+4v < 0$ which is the interior

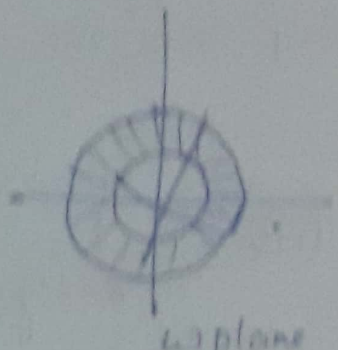
part of a circle

$y < 1/2 \quad \frac{-v}{u^2+v^2} < 1/2$

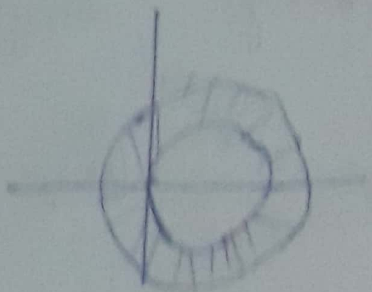
$-2v < u^2+v^2 \Rightarrow u^2+v^2+2v > 0$ exterior

part of a circle.





(2) $0 < y < 1/2$



$0 < y \rightarrow 0 < \frac{-v}{u^2+v^2} \quad 0 < -v \rightarrow v < 0$

$y < 1/2 \quad \frac{-v}{u^2+v^2} < 1/2 \Rightarrow -2v < u^2+v^2$

$u^2+v^2+2v > 0.$

exterior part of the circle passing through

Origin and centre (1,0)

Thus image of the region $0 < y < 1/2$ maps to

the exterior part of the circle below

U axis.

Linear fractional Transformation:

[Mobius transformations]

Linear fractional transformation are mappings

$W = \frac{az+b}{cz+d}$, $ad-bc \neq 0$, a, b, c, d are complex or real numbers.

Special Cases

$W = z+b$ $\longrightarrow$ Translation

$W = az$ with $|a|=1$ Rotation

$W = az+b$ linear transformation

$W = 1/z$ inversion.

Prop The mapping $w = \frac{1}{z}$ maps every straight line or circle onto a circle or straight line

Proof

$A(x^2+y^2) + Bx + Cy + D = 0 \rightarrow (1)$ A, B, C, D real or complex numbers

Represent straight line if $A = 0$.
Circle if $A \neq 0$.

we know that $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$.

$z\bar{z} = x^2+y^2$

(1) $\Rightarrow A z\bar{z} + B\left(\frac{z+\bar{z}}{2}\right) + C\left(\frac{z-\bar{z}}{2i}\right) + D = 0 \longrightarrow (2)$

By inversion $w = 1/z \Rightarrow z = 1/w$ $\bar{z} = 1/\bar{w}$

(ii) $\Rightarrow$

$$A \times \frac{1}{w} \cdot \frac{1}{\bar{w}} + B\left[\frac{\frac{1}{w}+\frac{1}{\bar{w}}}{2}\right] + C\left[\frac{\frac{1}{w}-\frac{1}{\bar{w}}}{2i}\right] + D = 0$$

$$\frac{A}{w\bar{w}} + B\left[\frac{\bar{w}+w}{2w\bar{w}}\right] + C\left[\frac{\bar{w}-\bar{w}}{2i\,w\bar{w}}\right] + D = 0$$

$\times\, w\bar{w}$

$$A + B\left[\frac{\bar{w}+w}{2}\right] + C\left[\frac{\bar{w}-w}{2i}\right) + D(w\bar{w}) = 0$$

$w = u + i\bar{v}$

$u = \frac{w+\bar{w}}{2}$ $\Longrightarrow$ $A + Bu - Cv + D(u^2+v^2) = 0$

$v = \frac{w-\bar{w}}{2i}$ $u^2+v^2 = w\bar{w}$ $D=0$ represent straight line

$D \neq 0$ " circle

Prob Find the image of the circle $|z-3| = 5$ under the transformation $w = \frac{1}{z}$

Ans $w = 1/z \Rightarrow z = 1/w \Rightarrow x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$

$x = \frac{u}{u^2+v^2}$ $y = \frac{-v}{u^2+v^2}$

$|z-3| = 5 \Rightarrow |x+iy-3| = 5 = \sqrt{(x-3)^2+y^2} = 5$

$(x-3)^2 + y^2 = 25$

$$\left[\frac{u}{u^2+v^2} - 3\right]^2 + \left[\frac{-v}{u^2+v^2}\right]^2 = 25$$

$$\frac{\left[u - 3(u^2+v^2)\right]^2 + v^2}{(u^2+v^2)^2} = 25$$

$\Rightarrow u^2 - 6u(u^2+v^2) + 9(u^2+v^2)^2 + v^2 = 25(u^2+v^2)^2$

$\Rightarrow u^2+v^2 - 6u(u^2+v^2) - 16(u^2+v^2)^2 = 0$

$\div\ u^2 v^2$.

$\Rightarrow\ 1 - 6u - 16(u^2+v^2) = 0 \Rightarrow$ Represent circle in w plane.

Fixed points (Problems).

1) Find the fixed points.

1) $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Ans fixed points $w = f(z) = z$

$\Rightarrow\ \frac{1}{2}\left(z + \frac{1}{z}\right) = z \Rightarrow z + \frac{1}{z} = 2z$

$z^2 + 1 = 2z^2 \Rightarrow z^2 - 1 = 0$

$z^2 = 1$

$z = \pm 1$.

2) $w = \dfrac{3z-4}{z-1}$

Ans: $z = \dfrac{3z-4}{z-1} \Rightarrow z^2 - z = 3z - 4$

$z^2 - 4z + 4 = 0$. $z = 2, 2$

3) $w = \dfrac{z-1}{z+1}$

Ans: $z = \dfrac{z-1}{z+1} \Rightarrow z^2 + z = z - 1 \Rightarrow z^2 + 1 = 0$

$z^2 = -1$ $z = \pm i$.

(4) $w = \dfrac{1}{z - 2i}$

Ans: $z = \dfrac{1}{z-2i} \Rightarrow z^2 - 2iz = 1$

$z^2 - 2iz - 1 = 0$. $= i, i$.

(5) $w = \dfrac{3iz + 13}{z - 3i}$ Home.

Discuss $w = \sin z$.

$f(z) = \sin z$ $\quad$ $f'(z) = \cos z$

$f'(z) = \cos z = 0$

$z = (2n+1)\pi/2$

$n = 0, \pm 1, \pm 2 \ldots$

The mapping is not conformal where $\cos z = 0$ i.e. $z = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$

$$w = \sin z$$
$$= \sin(x+iy)$$
$$= \sin x \cos iy + \cos x \sin iy$$
$$u + iv = \sin x \cosh y + i \cos x \sinh y$$

$u = \sin x \cosh y$ $\quad$ $v = \cos x \sinh y$

1) When $x = c$

$u = \sin c \cosh y$ $\quad$ $v = \cos c \sinh y$

$\cosh y = \dfrac{u}{\sin c}$ $\quad$ $\sinh y = \dfrac{v}{\cos c}$.

[We know that $\cosh^2 y - \sinh^2 y = 1$]

$$\Rightarrow \frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1$$ which is hyperbola

2) When $y = k$.

$u = \sin x \cosh k$ $\quad$ $v = \cos x \sinh k$.

$\sin x = \dfrac{u}{\cosh k}$ $\quad$ $\cos x = \dfrac{v}{\sinh k}$.

$\sin^2 x + \cos^2 x = 1 \Rightarrow \dfrac{u^2}{\cosh^2 k} + \dfrac{v^2}{\sinh^2 k}$ ellipse.

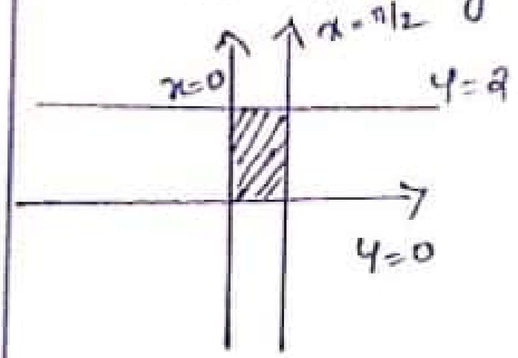
1 Find and sketch the image of the region
b $0 \le x \le \pi/2$ $0 < y < a$ under the transformation $W = \sin z$

Ans: $W = \sin z$

$u + iv = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y$ $\quad v = \cos x \sinh y$



$x = 0$ $\quad y = 0$

$x = \pi/2$ $\quad y = a$

$x = 0$ $\quad u = 0$ $\quad v = \sinh y$

the image of the line $x = 0$ is $u = 0$ (u' v axis)

$x = \pi/2$ $\quad u = \cosh y$ $\quad v = 0$ $\quad u \ge 1.$

$y = 0.$ $\quad u = \sin x$ $\quad v = 0$ $\quad -1 \le u \le 1$

$y = a$ $\quad$ Image. $\frac{u^2}{(\cosh a)^2} + \frac{v^2}{(\sinh a)^2} = 1$ ellipse

$$\frac{u^2}{\cosh^2 a} + \frac{v^2}{\sinh^2 a} < 1, \; u > 0, \; v > 0$$

Module - 4

Complex Variable Integration

Complex line integrals are of the form $\int_C f(z)dz$ or $\oint_C f(z)dz$. Here C is called the path of the integral.

<u>Simple curve</u> : A curve is simple if it doesnot intersect itself.

<u>Smooth Curve</u> : A curve 'c' has continuous and nonzero derivatives at each point then 'c' is called a smooth curve.

<u>Contour</u> : A contour is a piecewise smooth curve.

<u>Simply Connected domain</u>

A domain D is called simply connected, if every simple closed curve in D encloses only points of D.

<u>Properties of line integrals</u>

1) Linearity. $\int_C [k_1 f_1(z) + k_2 f_2(z)]dz = k_1 \int_C f_1(z)dz + k_2 \int_C f_2(z)dz$.

2) Sense Reversal : $\int_{z_0}^{z} f(z)dz = -\int_{z}^{z_0} f(z)dz$

(3) Partitioning of path.

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

Evaluation of line integrals

Method 1

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$.

Thm Let $f(z)$ be an analytic in a simple connected domain D. Then there exist an indefinite integral of $f(z)$ in the domain D, i.e. an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have.

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

Problems

1) Evaluate $\int_0^{1+i} z^2\,dz$.

Ans $\left.\frac{z^3}{3}\right]_0^{1+i} = \frac{(1+i)^3}{3} = \frac{-2+2i}{3} = -\frac{2}{3} + \frac{2}{3}i$

(2) $\int_{-\pi i}^{\pi i} \cos z\,dz$

Ans.. $\left[\sin z\right]_{-\pi i}^{\pi i} = \sin \pi i - \sin(-\pi i)$
$= 2\sin \pi i = 2\sinh \pi$

2) Evaluate $\int_0^{1+i} (x^2 - iy)\, dz$ along

(a) $y = x$ (b) $y = x^2$

(a) $y = x$ $dy = dx$ $x \to 0$ to 1

$$\int_0^1 (x^2 - ix)(dx + i\,dx)$$

$$= \int_0^1 (x^2 - ix)(1+i)\, dx = \int_0^1 x^2 + ix^2 - ix + x \, dx$$

$$= \left[\frac{x^3}{3} + i\frac{x^3}{3} - i\frac{x^2}{2} + \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{3} + \frac{i}{3} - \frac{i}{2} + \frac{1}{2} = \underline{\underline{\frac{5}{6} - \frac{1}{6}i}}$$

(b) Along $y = x^2$
$dy = 2x\, dx$ $x = 0$ to 1

$$\int_0^1 (x^2 - ix^2)(dx + i\,2x\,dx).$$

$$\int_0^1 (x^2 - ix^2)(1 + 2ix)\, dx$$

$$\int_0^1 x^2 + 2ix^3 - ix^2 + 2x^3\, dx.$$

$$\left[\frac{x^3}{3} + 2i\frac{x^4}{4} - i\frac{x^3}{3} + \frac{2x^4}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{i}{2} - i/3 + \frac{1}{2} = \underline{\underline{\frac{5}{6} + \frac{1}{6}i}}$$ Both are different

3) Evaluate $\int_C z^2\, dz$ where C is the line $x = 2y$ from $(0,0)$ to $(2,1)$.

Ans $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i\,2xy.$

$x = 2y \qquad dx = 2dy$

$f(z) = (2y)^2 - y^2 + i\,2\cdot 2y\cdot y$

$= 3y^2 + 4y^2 i$

$dz = dx + i\,dy \quad = 2dy + i\,dy$

$= (2+i)\,dy$

$$\int_C f(z)\,dz = \int_0^1 (3y^2 + 4y^2 i)(2+i)\,dy$$

$$= \int_0^1 6y^2 + 3y^2 i + 8y^2 i - 4y^2\,dy$$

$$= \left[\frac{6y^3}{3} + \frac{3y^3}{3} i + \frac{8y^3}{3} i - \frac{4y^3}{3}\right]_0^1$$

$$= 2 + i + 8/3\, i - 4/3$$

$$= \frac{2}{3} + \frac{11}{3} i$$

Method 2 — Second evaluation method

This method is not restricted to analytic functions but applies to any continuous complex function.

Let C be a piecewise smooth path represented by $z = z(t)$ where $a \le t \le b$. Let $f(z)$ be continuous function on C. Then

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt$$

$$\left\{ \dot{z}(t) = \frac{dz}{dt} \right.$$

Problem

1) Evaluate $\int_C \frac{dz}{z}$ where C is the unit circle in anticlockwise direction

Ans. C is the unit circle its parametric representation $z(t) = e^{it}$ $0 \le t \le 2\pi$.

$$\dot{z}(t) = ie^{it}$$

$$f(z) = \frac{1}{z} \quad f(z(t)) = \frac{1}{e^{it}} = e^{-it}$$

$$\int_C f(z)dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt$$

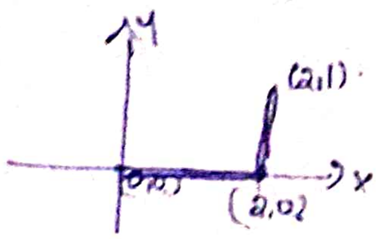
$$= \int_0^{2\pi} e^{-it} \cdot ie^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$$

Result $\int_{|z|=1} \frac{dz}{z} = 2\pi i$

2) evaluate $\int_C e^z dz$ where C is the shortest path from πi to $2\pi i$

Ans $\int_{\pi i}^{2\pi i} e^z dz = \left[e^z\right]_{\pi i}^{2\pi i} = e^{2\pi i} - e^{\pi i} = e^{\pi i}(e^{\pi i} - 1)$

3) Evaluate $\int_C z^2 dz$, where C is given by line along the real axis from $(0,0)$ to $(2,0)$ and then vertically to $(2,1)$

Ans 

Take the path from $(0,0)$ to $(2,0)$ as C_1 and path from $(2,0)$ to $(2,1)$ as C_1

$$\int_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} z^2 dz$$

$\underline{C_1}$ $\quad y=0 \quad dy=0 \qquad z^2 = x^2 - y^2 + i\,2xy = x^2$

x varies from 0 to 2

$$\int_{C_1} z^2 dz = \int_0^2 x^2 dx = \frac{x^3}{3}\Big]_0^2 = \underline{\underline{8/3}}$$

$\underline{C_2}$ $\quad x=2 \quad dx=0 \qquad z^2 = 4 - y^2 + i\,4y$

$y \to 0$ to 1

$$\int_{C_2} z^2 dz = \int_0^1 (4 - y^2 + i\,4y)\,i\,dy = i\left(4y - \frac{y^3}{3} + i\,\frac{4y^2}{2}\right]_0^1$$

$$= \left(4 - \frac{1}{3} + 2i\right)i$$

$$= \left(\frac{11}{3} + 2i\right)i = \underline{\underline{\frac{11}{3}i - 2}}$$

$$\int_C z^2 dz = \frac{8}{3} + \frac{11}{3}i - 2 = \underline{\underline{\frac{2}{3} + \frac{11}{3}i}}$$

(4). Evaluate $\int_C \bar{z}\,dz$ where C is parametrized by $z(t) = 3t + it^2, \ -1 \le t \le 4$

Ans

$z = 3t + it^2$

$\bar{z} = 3t - it^2 \qquad z'(t) = (3 + 2it)$

$$\int_C \bar{z}\,dz = \int_C (3t - it^2)(3 + 2it)\,dt.$$

$$= \int_C 9t + 6it^2 - 3it^2 + 2t^3 dt.$$

$$= \frac{9t^2}{2} + \frac{3it^3}{3} + \frac{2t^4}{4}\Big]_{-1}^{4}.$$

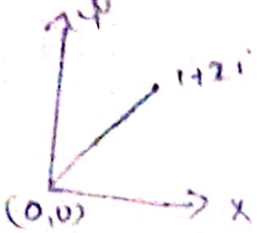
$$= \cancel{4}72 + 64i + 128 - (9/2 - i + 1/2).$$

$$= 200 + 64i - 5 + i = \underline{\underline{195 + 65i}}$$

(5) Evaluate $\int_0^{1+2i} f(z)\,dz$ when $f(z) = \mathrm{Re}(z)$ (4)

(i) along a straight line from 0 to $1+2i$

Ans



(ii) along the real axis from 0 to 1 and then along a line parallel to imaginary axis from $z=1$ to $z=1+2i$.

Ans

(1) 0 to $1+2i$

$x_1, y_1 \to (0,0)$ $\qquad x_2, y_2 \to 1, 2.$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \implies \frac{x-0}{1-0} = \frac{y-0}{2-0}$$

$$\implies x = y/2 \qquad \underline{y = 2x}$$

$$dy = 2\,dx.$$

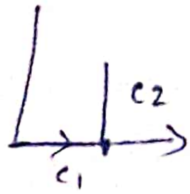
$x \to 0$ to 1 $\qquad dz = dx + i\,dy = dx + i\,2dx$

$$= (1+2i)\,dx.$$

$$\int_C \mathrm{Re}(z)\,dz = \int_0^1 x(1+2i)\,dx = (1+2i)\left.\frac{x^2}{2}\right]_0^1$$

$$= \underline{\underline{\frac{1}{2} + i}}$$

(ii)



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

Along C_1 $\qquad y = 0 \quad dy = 0 \quad dz = dx \quad x \to 0$ to 1

$$\int_{C_1} f(z)\,dz = \int_0^1 x\,dx = \left.\frac{x^2}{2}\right]_0^1 = \underline{\underline{\frac{1}{2}}}$$

Along C_2 $\qquad x = 1 \quad dx = 0 \quad dz = i\,dy \quad y \to 0$ to 2

$$\int_{C_2} f(z)\,dz = \int_0^2 x\,dz = \int_0^2 1.\, i\,dy = (iy)_0^2 = \underline{\underline{2i}}$$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = \underline{\underline{\frac{1}{2} + 2i}}$$

(6) Evaluate $\int_0^{2+i} (\bar{z})^2 \, dz$ along the line $y = x/2$

Ans $\bar{z} = x - iy \qquad (\bar{z})^2 = (x-iy)^2 = x^2 - y^2 - i2xy$

$y = x/2 \qquad x = 2y \qquad dx = 2dy \qquad dz = dx + i\,dy$

$y \to 0 \text{ to } 1 \qquad = 2dy + i\,dy$

$= (2+i)\,dy$

$\int_0^1 \left((2y)^2 - y^2 - i\,2 \times 2y \times y\right)(2+i)\,dy$

$\int_0^1 (3y^2 - 4y^2 i)(2+i)\,dy$

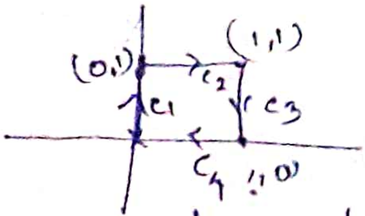
$= \int_0^1 6y^2 + 3y^2 i - 8y^2 i + 4y^2 \, dy$

$\int_0^1 10y^2 - 5y^2 i \, dy = \frac{10y^3}{3} - \frac{5y^3}{3} i \Big]_0^1$

$= \frac{10}{3} - \frac{5}{3} i$

(7). Evaluate $\oint_C f(z)\,dz$, $f(z) = \operatorname{Re}(z^2)$ of the boundary C of the square with vertices $(0, i, 1+i, 1)$ clockwise.

Ans



$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \int_{C_4} f(z)\,dz$

$\operatorname{Re}(z^2) = \underline{x^2 - y^2}$

$z^2 = x^2 - y^2 + i2xy$

$\underline{C_1} \quad x = 0 \quad dx = 0 \quad \operatorname{Re}(z^2) = -y^2 \quad dz = i\,dy \quad y \to 0 \text{ to } 1$

$\int_{C_1} \operatorname{Re}(z^2)\,dz = \int_0^1 -y^2 \cdot i\,dy = -i\frac{y^3}{3}\Big]_0^1 = \underline{-i/3}$

$\int_{C_2} \operatorname{Re}(z^2)\,dz = \int_0^1 (x^2 - 1)\,dx = \frac{x^3}{3} - x\Big]_0^1 = \frac{1}{3} - 1 = \underline{-2/3}$

$\int_{C_3} \operatorname{Re}(z^2)\,dz = \int_1^0 (1-y^2)\,i\,dy = \left[i\left(y - \frac{y^3}{3}\right)\right]_1^0 = -\frac{2i}{3}$

$\int_{C_4} \operatorname{Re}(z^2)\,dz = \int_1^0 x^2\,dx = \frac{x^3}{3}\Big]_1^0 = -1/3 \quad \therefore \int_C f(z)\,dz = \underline{-1 - i}$

clockwise direction

Cauchy integral theorem

If $f(z)$ is analytic in a simply connected domain D then for every simple closed path 'C' in D $\oint_C f(z)\,dz = 0$

* If $f(z)$ is analytic in a simply connected domain D then the integral of $f(z)$ is independent of path in D.

Pbms

1) $\oint_C \frac{1}{z^2+1}\,dz \qquad C: |z| = \frac{1}{2}$.

$z^2+1=0 \Rightarrow z^2=-1$

$z=\pm i$ are singular pts

$\therefore f(z)$ is not analytic at $z=+i$ and $z=-i$

$|z| = \frac{1}{2} \qquad z=i \qquad |z|=1>\frac{1}{2}$ outside c

$\qquad\qquad z=-i \qquad |z|=1>\frac{1}{2}$ outside c.

$\therefore i, -i$ lies outside c

$\therefore f(z)$ analytic at all pts inside $|z|=\frac{1}{2}$

$\therefore$ By Cauchy integral thm $\oint_C f(z)\,dz = 0$

$$\oint_C \frac{1}{z^2+1}\,dz = 0.$$

(2) Cauchy Integral formula

Let $f(z)$ be an analytic function in a simply connected domain D, then for any point z_0 in D and any closed path C in D that encloses z_0 (integration taken in anti clockwise direction)

$$\oint \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

Also $\oint_c \frac{f(z)\,dz}{(z-z_0)^2} = 2\pi i\, f'(z_0)$

$\oint_c \frac{f(z)}{(z-z_0)^3}\,dz = \frac{2\pi i}{2!} f''(z_0)$

In general $\oint_c \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)$

Pbms

1 Evaluate $\oint_C \frac{\cos \pi z}{z-2}\,dz$ over C $|z| = 3$

Ans $z-2=0 \Rightarrow z=2$ singular point

given $|z| = 3$ $|2| \leq 3$ lies inside c.

By Cauchy Integral formula

$\int_c \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$

Here $f(z) = \cos \pi z$

$f(z_0) = f(2) = \cos 2\pi = 1$

$= 2\pi i \times 1$

$= 2\pi i$

2 $\oint_c \frac{e^z}{z+1}\,dz$ (i) $c: |z| = 2$

(ii) $c: |z| = \frac{1}{2}$

(iii) $c: |z+1| = 1/2$

$z+1=0 \Rightarrow z=-1$ singular point

(i). $|z| = 2$, $|-1| \leq 2$ inside c.

By C.I.F $\int \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$

$\int \frac{e^z}{z+1}\,dz = 2\pi i\, f(-1)$

$= \frac{2\pi i}{e}$

$f(z) = e^z$

$f(-1) = e^{-1}$

(ii) $|z| = \frac{1}{2}$ $|-1| > 1/2$ outside c.

By C.I.T $\oint_c f(z)\,dz = 0 \Rightarrow \oint_c \frac{e^z}{z+1}\,dz = 0$

(iii) $|z+1| = 1/2$.

$|-1+1| = 0 < 1/2$ inside c

By C.I.F $\oint \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

$= 2\pi i\, f(-1) = \frac{2\pi i}{4}$

③ Evaluate $\oint_C \frac{z+2}{z-2} dz$ where C is the circle $|z-1| = 2$.

Ans: $z-2=0 \Rightarrow z=2$ singular pt.

$|z-1| = 2 \Rightarrow |2-1| < 2$ inside c.

By C.I.F $\int \frac{z+2}{z-2} dz = 2\pi i f(2)$ $\quad f(z) = z+2$

$= 2\pi i \times 4 = \underline{8\pi i}$ $\quad f(2) = 2+2 = 4$

④ $\oint_C \frac{\sin 2z}{z^4} dz \qquad C: |z| = 1$

$z=0$ is a singular pt of order 4

$|z| = 1 \Rightarrow |0| < 1$ inside c

$\therefore$ By C.I.F $\oint_C \frac{\sin 2z\, dz}{z^4} = \frac{2\pi i}{3!} f'''(0)$

$= \frac{2\pi i}{6} \times -8\cos 2.0$ $\quad f(z) = \sin 2z$

$= \underline{\frac{-8\pi i}{3}}$ $\quad f'(z) = 2\cos 2z$, $\quad f''(z) = -4\sin 2z$, $\quad f'''(z) = -8\cos 2z$

⑤. $\oint_C \frac{z^2+5z+3}{(z-2)^2} dz \qquad C: |z| = 3$

$(z-2)^2 = 0 \Rightarrow z = 2$ pole of order 2

By C.I.F $\oint \frac{f(z)}{(z-z_0)^2} = 2\pi i f'(z_0)$ $\quad f(z) = z^2+5z+3$

$= 2\pi i\, f'(2)$ $\quad f'(z) = 2z+5$

$= 2\pi i \times 9$ $\quad f'(2) = 2 \times 2 + 5$

6 $\oint_c \frac{e^z}{z(1-z)^3} dz \quad |z| = 1/2$

$z(1-z)^3 = 0 \Rightarrow z = 0$ of order 1, $z = 1$ of order 3

$|z| = 1/2 \Rightarrow |0| < 1/2$ inside c

$|1| > 1/2$ out side c

$$\oint_c \frac{e^z}{z(1-z)^3} dz = \oint_c \frac{e^z/(1-z)^3}{z} dz = 2\pi i\, f(0).$$

$$= \underline{2\pi i}$$

$f(z) = \frac{e^z}{(1-z)^3}$

$f(0) = \frac{1}{1} = 1$

(7) $\oint_c \frac{z^2+2z+3}{z^2-1} dz \quad c: |z-1| = 1$

$z^2-1 = 0 \Rightarrow z = \pm 1.$

$|z-1| = 1 \Rightarrow |1-1| < 1$ inside c

$|-1-1| > 1$ out side c

$$\oint_c \frac{z^2+2z+3}{(z+1)(z-1)} dz = \oint_c \frac{z^2+2z+3/z+1}{z-1} dz$$

$= 2\pi i\, f(1)$

$= 2\pi i \times 3$

$= 6\pi i$

$f(z) = \frac{z^2+2z+3}{z+1}$

$f(1) = \frac{6}{2} = 3$

(8). Using Cauchy's formula $\oint_c \frac{z+1}{z^4+2iz^3} dz \quad c: |z| = 1$

Ans $z^4 + 2iz^3 = 0 \Rightarrow z^3(z+2i) = 0 \Rightarrow z = 0$ of order 3

$z = -2i$

$|z| = 1 \quad |0| < 1$ inside c.

$|-2i| > 1$ out side c.

$$\oint_c \frac{z+1}{z^4+2iz^3} dz = \oint_c \frac{z+1/z+2i}{z^4} dz \cdot \frac{2\pi i}{3!} f'''(-2i)$$

$= \frac{\pi i}{2} \cdot \frac{\pi}{4}$

(9) Evaluate $\oint_C \frac{dz}{z-3i}$, C is the circle $|z| = \pi$ counter clockwise.

→ $z-3i = 0 \Rightarrow z = 3i$ singularity

$|z| = \pi \Rightarrow |3i| = 3 < \pi$ inside c.

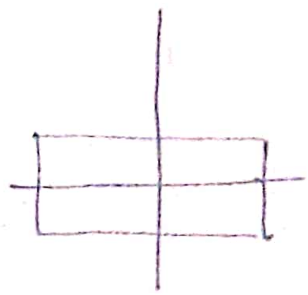
∴ By cauchy integral formula

$$\oint_C \frac{dz}{z-3i} = 2\pi i f(3i) = \underline{\underline{2\pi i}}$$

$f(z) = 1$
$f(3i) = 1$

10 Evaluate $\oint_C \frac{\cos \pi z}{z^2-1}$ where C is the rectangle with vertices $2 \pm i, -2 \pm i$

→ $z^2-1 = 0 \Rightarrow z^2 = 1$
$z = \pm 1$



$z = 1$ inside c
$z = -1$ inside c

$$\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$$
$$= \frac{A(z-1)+B(z+1)}{(z+1)(z-1)}$$

$A(z-1) + B(z+1) = 1$

$z = 1 \Rightarrow 2B = 1 \quad B = 1/2$

$z = -1 \quad -2A = 1 \quad A = -1/2$

$$\oint_C \frac{\cos \pi z}{z^2-1} = -1/2 \oint \frac{\cos\pi z\, dz}{z+1} + \frac{1}{2} \oint_C \frac{\cos \pi z\, dz}{z-1}$$
$$= -\frac{1}{2} 2\pi i f(-1) + \frac{1}{2} 2\pi i f(1)$$
$$= -\pi i \times -1 + \pi i \cdot -1 = \underline{0}$$

11 $\oint_c \frac{\cos\pi z^2 + \sin\pi z^2 \, dz}{(z-1)(z-3)}$ where $c. |z| = \pi$.

$\rightarrow$ $(z-1)(z-3)=0 \implies z=1$ & $z=3$

$z=1$ $|1|<\pi$ inside c, $z=3$ $|3|<\pi$ inside c.

$$\frac{1}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$
$$= \frac{A(z-3)+B(z-1)}{(z-1)(z-3)}$$

$A(z-3)+B(z-1) = 1$

$z=3$ $2B=1$ $B = 1/2$.

$z=1$ $-2A=1$ $A=-1/2$.

$$\frac{\cos\pi z^2 + \sin\pi z^2}{(z-1)(z-3)} = -\frac{1}{2}\frac{\cos\pi z^2+\sin\pi z^2}{z-1} + \frac{1}{2}\frac{\cos\pi z^2+\sin\pi z^2}{z-3}$$

$$\oint_c \frac{\cos\pi z^2+\sin\pi z^2}{(z-1)(z-3)}dz = -\frac{1}{2}\int\frac{\cos\pi z^2+\sin\pi z^2}{z-1}dz + \frac{1}{2}\int\frac{\cos\pi z^2+\sin\pi z^2}{z-3}dz$$

$$= -\frac{1}{2}2\pi i\, f(1) + \frac{1}{2}2\pi i\, f(3)$$

$$= -\pi i(\cos\pi + \sin\pi) + \pi i[\cos 9\pi + \sin 9\pi]$$

$$= \pi i - \pi i = \underline{0}$$

12 Evaluate $\oint_c \frac{4-3z}{z(z-1)(z-2)}dz$ $C: |z|=3$

$\rightarrow$ $z=0,1,2$ $z=0$ inside c $z=1$ $|1|<3$ inside c
$|2|<3$ inside c

$$\frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$ solving $A = 1/2$ $B=-1$ $C=1/2$

$$\oint_c \frac{4-3z}{z(z-1)(z-2)} = \frac{1}{2}\int\frac{4-3z}{z}dz - \int\frac{4-3z}{z-1}dz + \frac{1}{2}\int\frac{4-3z}{z-2}dz$$

$$= \frac{1}{2}\times 2\pi i\times 4 - 2\pi i\times 1 + \frac{1}{2}2\pi i\times -2$$
$$=0$$

13 Evaluate $\oint_c \frac{4-3z}{z(z-1)(z-2)}$ $\quad C: |z| = 3/2$. ⑧

→ $z = 0, 1, 2$ singular pts.

$z=0$ inside c

$z=1$ inside c $\quad \oint_c \frac{4-3z/z-2}{z(z-1)} dz$

$z=2$ outside c

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} = \frac{A(z-1)+Bz}{z(z-1)}$$

$1 = A(z-1) + Bz$ $\quad z=0 \quad A=-1$

$\quad z=1 \quad B=1$

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$$\oint_c \frac{4-3z/z-2}{z(z-1)} dz = -\oint_c \frac{4-3z/z-2}{z} + \oint_c \frac{4-3z/z-2}{z-1} dz$$

$= -2\pi i\, f(0) + 2\pi i\, f(1)$

$f(z) = \frac{4-3z}{z-2}$ $\quad = -2\pi i \times -2 + 2\pi i \times -1$

$f(0) = \frac{4}{-2} = -2$ $\quad = 4\pi i - 2\pi i = \underline{2\pi i}$

$f(1) = \frac{1}{-1} = -1$

14 $\oint_c \frac{3z-1}{z^3-z} dz$ $\quad$ (i) $C: |z| = \frac{1}{2}$

(ii) $C: |z| = 2$

→ $z^3 - z = 0 \Rightarrow z(z^2-1) = 0 \Rightarrow z = 0, -1, 1$ singular pt

$\Rightarrow z(z+1)(z-1) = 0$

(i) $C: |z| = 1/2$

$z=0$ inside c $\quad z=-1$ outside c $\quad z=1$ outside c

$$\oint_c \frac{3z-1}{z(z-1)(z+1)} dz = \oint_c \frac{3z-1/(z-1)(z+1)}{z} dz$$

$= 2\pi i\, f(0)$ $\quad f(z) = 3z-1/(z-1)(z+1)$

$= \underline{2\pi i}$ $\quad f(0) = \frac{-1}{-1 \times 1} = 1$

(ii) $|z| = 2$

$z=0$ inside c $\quad z=-1$ inside c $\quad z=1$ inside c

$$\frac{1}{z(z-1)(z+1)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+1} = \frac{A(z-1)(z+1) + Bz(z+1) + Cz(z-1)}{z(z-1)(z+1)}$$

$$1 = A(z-1)(z+1) + Bz(z+1) + Cz(z-1)$$

$z = 0 \qquad 1 = -A \qquad A = -1$

$z = 1 \qquad 1 = 2B \qquad B = 1/2$

$z = -1 \qquad 1 = 2C \qquad \underline{C = 1/2}$

$$\oint_C \frac{3z-1}{z(z-1)(z+1)}\,dz = -\int \frac{3z-1}{z}\,dz + \frac{1}{2}\int \frac{3z-1}{z-1}\,dz + \frac{1}{2}\int \frac{3z-1}{z+1}\,dz$$

$$= -2\pi i\, f(0) + \frac{1}{2}\, 2\pi i\, f(1) + \frac{1}{2}\, 2\pi i\, f(-1)$$

$f(z) = 3z-1$

$f(0) = -1$

$f(1) = 2$

$f(-1) = -4$

$$= -2\pi i \times -1 + \pi i \times 2 + \pi i \times -4$$

$$\underline{= 0}$$

15 $\displaystyle\int_C \frac{z^2}{z^2-1}\,dz \qquad C: |z-1-i| = \pi/2$

$\rightarrow \quad z^2 - 1 = 0 \qquad z = \pm 1$

$C: |z-1-i| =$

$z = 1 \quad |1-1-i| = |-i| = 1 < \pi/2$ inside c

$z = -1 \quad |-1-1-i| = |-2-i| > \pi/2$ outside C

$$\oint_C \frac{z^2}{(z+1)(z-1)}\,dz = \oint_C \frac{z^2/z+1}{z-1}\,dz$$

$$= 2\pi i\, f(1)$$

$f(z) = z^2/z+1$

$f(1) = \underline{\frac{1}{2}}$

$$= 2\pi i \times 1/2$$

$$\underline{= \pi i}$$

Taylor Series, Maclaurin Series

The Taylor Series of an analytic function $f(z)$ inside a circle with centre z_0 is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where } a_n = \frac{1}{n!} f^{(n)}(z_0)$$

or $$f(z) = f(z_0) + \frac{(z-z_0)}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots$$

If $z_0 = 0$ then Taylor series

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) \cdots$$ is called

Maclaurin series

Note:

$$(1+z)^{-1} = 1 - z + z^2 \cdots \quad \sum_{n=0}^{\infty} (-1)^n z^n$$

$$(1-z)^{-1} = 1 + z + z^2 \cdots = \sum_{n=0}^{\infty} z^n$$

$$(1+z)^{-2} = 1 - 2z + 3z^2 \cdots = \sum_{n=0}^{\infty} (-1)^n (n+1) z^n$$

$$(1-z)^{-2} = 1 + 2z + \cdots = \sum_{n=0}^{\infty} (n+1) z^n$$

These expansions are valid if $|z|<1$

Pbms

1. Expand $f(z) = \dfrac{1}{z+2}$ at $z=1$ as a Taylor Series

$$\rightarrow \frac{1}{z+2} = \frac{1}{z-1+3} = \frac{1}{3\left[1+\frac{z-1}{3}\right]} = \frac{1}{3}\left[1+\frac{z-1}{3}\right]^{-1}$$

$$= \frac{1}{3}\left[1 - \left(\frac{z-1}{3}\right) + \left(\frac{z-1}{3}\right)^2 \cdots\right]$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{3}\right)^n$$

Find the Taylor series $f(z) = \dfrac{1}{z^2-z-6}$ about $z=-1$

$$\frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2}.$$

$$= \frac{A(z+2)+B(z-3)}{(z-3)(z+2)}$$

$$1 = A(z+2)+B(z-3)$$

$z=3$ $\quad 5A=1 \quad A = 1/5$

$z=-2$ $\quad -5B=1 \quad B = -1/5$.

$$\frac{1}{z^2-z-6} = \frac{1/5}{z-3} - \frac{1/5}{z+2}.$$

$$\frac{\frac{1}{5}}{z+1-4} - \frac{\frac{1}{5}}{z+1+1} = \frac{1}{5}\left[\frac{1}{-4+(z+1)} - \frac{1}{1+(z+1)}\right]$$

$$= \frac{1}{5}\left[\frac{1}{-4\left[1-\left(\frac{z+1}{4}\right)\right]} - \frac{1}{1+(z+1)}\right]$$

$$= \frac{1}{5}\left\{\frac{1}{-4}\left[1-\left(\frac{z+1}{4}\right)\right]^{-1}\right\} - \left[1+(z+1)\right]^{-1}$$

$$= \frac{-1}{20}\left\{1 + \frac{z+1}{4} + \left(\frac{z+1}{4}\right)^2 + \cdots\right] - \frac{1}{5}\left\{1-(z+1)+(z+1)^2 \cdots\right\}$$

$$= \frac{-1}{20}\sum_{n=0}^{\infty}\left(\frac{z+1}{4}\right)^n - \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n (z+1)^n$$

3) Find Maclurian series of $f(z) = \sin z$

$\rightarrow$ $f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \cdots$

$$= \frac{z}{1!}\times 1 + \frac{z^3}{3!}\times -1 + \frac{z^5}{5!}\times 1 \cdots$$

$$= \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$f(z) = \sin z$

$f(0) = 0$

$f'(z) = \cos z$, $f'(0) = 1$

$f''(z) = -\sin z$ $f''(0) = 0$

$f'''(z) = -\cos z$ $f'''(0) = -1$.

4 Write $f(z) = \sin z$ as a Taylor Series about $z = \pi/4$

$\rightarrow$ $f(z) = \sin z$ $\qquad f(\pi/4) = \sin \pi/4 = 1/\sqrt{2}$

$f'(z) = \cos z$ $\qquad f'(\pi/4) = \cos \pi/4 = 1/\sqrt{2}$

$f''(z) = -\sin z$ $\qquad f''(\pi/4) = -\sin \pi/4 = -1/\sqrt{2}$

Taylor Series $f(z) = f(z_0) + \frac{z - z_0}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0)$

$$= \frac{1}{\sqrt{2}} + \frac{z - \pi/4}{1!} \, 1/\sqrt{2} + \frac{(z - \pi/4)^2}{2!} \times -1/\sqrt{2} \ldots$$

$$= \frac{1}{\sqrt{2}} \left[1 + \frac{z - \pi/4}{1} - \frac{(z - \pi/4)^2}{2} \ldots \right]$$

5 Find the Maclaurin Series of $f(z) = \frac{1}{1+z^2}$

$\rightarrow$ $$\frac{1}{1+z^2} = (1+z^2)^{-1} = 1 - z^2 + z^4 + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

Some important Taylor series (Maclaurin series)

1. Geometric Series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 \ldots$$

(2) Exponential Series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \ldots$$

(3) Trignometric Series

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \ldots$$

(4) Hyperbolic Series

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

(5) Logarithemic Series

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \cdots$$

$$-\ln(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} \cdots$$

$$\ln \frac{1+z}{1-z} = 2\left[z + \frac{z^3}{3} + \frac{z^5}{5} \cdots \right]$$

Module 5 – Residue Integration

Laurent Series

Laurent Series generalise Taylor Series.
If in an application, we want to develop a function $f(z)$ in powers of $z-z_0$ when $f(z)$ is singular at z_0, we cannot use a Taylor Series. Instead we use a new kind of series, called Laurent Series consisting of positive integer powers of $z-z_0$ (and a constant) as well as negative integer powers of $z-z_0$.

ie $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

$= a_0 + a_1(z-z_0) + \dots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots$

Pbms

1 Expand $f(z) = \frac{1}{z-z^3}$ in Laurent Series for the region $1 < |z+1| < 2$

→ $1 < |z+1| < 2 \implies 1 < |z+1|$ ie $\frac{1}{|z+1|} < 1$

$|z+1| < 2 \implies \left|\frac{z+1}{2}\right| < 1$

$f(z) = \frac{1}{z-z^3} = \frac{1}{z(1-z^2)} = \frac{1}{z(1+z)(1-z)}$

$\frac{1}{z(1+z)(1-z)} = \frac{A}{z} + \frac{B}{1+z} + \frac{C}{1-z}$

$= \frac{A(1+z)(1-z) + Bz(1-z) + Cz(1+z)}{z(1+z)(1-z)}$

$1 = A(1+z)(1-z) + Bz(1-z) + Cz(1+z)$

Solving we get $A = +1$ $B = -1/2$ $C = 1/2$.

$$\frac{1}{z-z^3} = \frac{1}{z} + \frac{-1/2}{1+z} + \frac{1/2}{1-z}$$

$$= \frac{1}{-1+(z+1)} + \frac{-\frac{1}{2}}{z+1} + \frac{1/2}{2-(z+1)}$$

$$= \frac{+1}{-1+(z+1)} - \frac{\frac{1}{2}}{z+1} + \frac{\frac{1}{2}}{2\left(1-\frac{z+1}{2}\right)}$$

$$= \frac{1}{(z+1)\left(1-\frac{1}{z+1}\right)} - \frac{1/2}{z+1} + \frac{1}{4}\left(1-\frac{z+1}{2}\right)^{-1}$$

$$= \frac{1}{(z+1)}\left[1-\frac{1}{z+1}\right]^{-1} - \frac{1}{2(z+1)} + \frac{1}{4}\left[1-\frac{z+1}{2}\right]^{-1}$$

$$= \frac{1}{(z+1)}\left\{1-\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^2\cdots\right] - \frac{1}{2(z+1)} \text{ principal part}$$

$$+ \frac{1}{4}\left\{1+\left(\frac{z+1}{2}\right)+\left(\frac{z+1}{2}\right)^2+\cdots\right]$$

2) Expand $f(z) = \frac{z}{(z+1)(z+2)}$ in Laurent series $z=-2$.

$\longrightarrow$ $f(z) = \frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2} = \frac{A(z+2)+B(z+1)}{(z+1)(z+2)}$

$z = A(z+2)+B(z+1)$

$A = -1 \qquad B = 2.$

$$f(z) = \frac{-1}{z+1} + \frac{2}{z+2}.$$

$$= \frac{-1}{(z+2)-1} + \frac{2}{z+2}$$

$$= \frac{-1}{-1+(z+2)} + \frac{2}{z+2)}$$

$$= \frac{1}{1-(z+2)} + \frac{2}{z+2} \Rightarrow \left[1-(z+2)\right]^{-1} + \frac{2}{z+2}$$

$$= 1+(z+2)+(z+2)^2 \cdots$$

$$+ \frac{2}{z+2} \text{ Princ.}$$

3 Find the Laurent series of $\frac{1}{z^3 - z^4}$ about $z=0$

$\rightarrow$ $f(z) = \frac{1}{z^3(1-z)} = \frac{1}{z^3}(1-z)^{-1}$

$= \frac{1}{z^3}\{1+z+z^2 \ldots\}$

$= \underbrace{\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z}}_{\text{Principal part}} + 1 + z + \ldots$

4 Find Laurent series of $z^2 e^{1/z}$ about $z=0$.

$\rightarrow$ $z^2 e^{1/z} = z^2 \left[1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots\right)$

$= z^2 + \frac{z}{1!} + \frac{1}{2!} + \frac{1}{3!z} + \ldots$

5 Expand $f(z) = \frac{z-1}{z^2 - 5z + 6}$ in $2 < |z| < 3$ as a Laurent series

$\frac{z-1}{z^2-5z+6} = \frac{z-1}{(z-2)(z-3)}$

$= \frac{A}{z-2} + \frac{B}{z-3} = \frac{A(z-3)+B(z-2)}{(z-2)(z-3)}$

$z-1 = A(z-3) + B(z-2)$

$z=2 \qquad 1 = -A \qquad A = -1$

$z=3 \qquad 2 = B \qquad B = 2.$

$2 < |z| < 3$

$\frac{z-1}{z^2-5z+6} = \frac{-1}{z-2} + \frac{2}{z-3}$

$2 < |z| \Rightarrow \frac{2}{|z|} < 1$

$|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$

$= \frac{-1}{z(1-\frac{2}{z})} + \frac{2}{-3[1-\frac{z}{3}]}$

$= -\frac{1}{z}(1-\frac{2}{z})^{-1} - \frac{2}{3}(1-\frac{z}{3})^{-1}$

$$-\frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]-\frac{2}{3}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right]$$

6 Find Laurent Series of $f(z)=\frac{-2z+3}{z^2-3z+2}$ with Centre 0 in ① $|z|<1$ ② $1<|z|<2$.

→ $f(z)=\frac{-2z+3}{z^2-3z+2}=\frac{-2z+3}{(z-1)(z-2)}=\frac{A}{(z-1)}+\frac{B}{z-2}$

$-2z+3=A(z-2)+B(z-1) \Rightarrow A=-1 \quad B=-1$

$f(z)=\frac{-1}{z-1}+\frac{-1}{z-2}.$

(1) $|z|<1 \quad |\frac{z}{2}|<1.$

$$=\frac{-1}{-1+z}-\frac{1}{-2+z}=\frac{1}{1-z}+\frac{1}{2(1-z/2)}$$

$$=(1-z)^{-1}+\frac{1}{2}(1-z/2)^{-1}$$

$$=1+z+z^2\cdots+\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]$$

$$1+z+z^2+\cdots+\frac{1}{2}+\frac{z}{4}+\frac{z^2}{8}+\cdots$$

$\frac{3}{2}nt$

(2). $1<|z|<2 \Rightarrow 1<|z| \Rightarrow \frac{1}{|z|}<1$

$|z|<2 \Rightarrow |\frac{z}{2}|<1$

$$f(z)=-\frac{1}{z-1}-\frac{1}{z-2}.$$

$$=-\frac{1}{z(1-1/z)}-\frac{1}{-2+z}=-\frac{1}{z}(1-1/z)^{-1}+\frac{1}{2(1-z/2)}$$

$$=-\frac{1}{z}(1-1/z)^{-1}+\frac{1}{2}(1-z/2)^{-1}.$$

$$=-\frac{1}{z}\left(1+1/z+(1/z)^2\cdots\right]+\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right.$$

$$=-\frac{1}{z}-1/z^2-\frac{1}{z^3}\cdots+\frac{1}{2}\left[1+z/2+(z/2)^2\cdots\right.$$

Expand $f(z) = \dfrac{z^2-1}{z^2-5z+6}$ in $2<|z|<3$

$\rightarrow$

$$f(z) = \frac{z^2-1}{z^2-5z+6} = 1 + \frac{5z-7}{z^2-5z+6}$$

$$\begin{array}{r|l} z^2-5z+6 & z^2+0z-1 \quad (1 \\ & z^2-5z+6 \\ \hline & 5z-7 \end{array}$$

$$\frac{5z-7}{z^2-5z+6} = \frac{5z-7}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3} = \frac{A(z-3)+B(z-2)}{(z-2)(z-3)}$$

$$5z-7 = A(z-3)+B(z-2)$$

$z=2 \quad 3 = -A \qquad A=-3$

$z=3 \quad 8 = B \qquad B=8$

$$\frac{5z-7}{(z-2)(z-3)} = \frac{-3}{z-2} + \frac{8}{z-3}$$

$2<|z|<3 \Rightarrow 2<|z| \qquad \dfrac{2}{|z|}<1$

$|z|<3 \qquad \left|\dfrac{z}{3}\right|<1$

$$\frac{5z-7}{(z-2)(z-3)} = \frac{-3}{z\left(1-\frac{2}{z}\right)} + \frac{8}{-3\left(1-\frac{z}{3}\right)}$$

$$= -\frac{3}{z}\left[1-\frac{2}{z}\right]^{-1} - \frac{8}{3}\left(1-\frac{z}{3}\right)^{-1}$$

$$= -\frac{3}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2 \cdots\right] - \frac{8}{3}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right]$$

$$= -\frac{3}{z} - \frac{6}{z^2} \cdots - \frac{8}{3}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right]$$

$$f(z) = 1 + -\frac{3}{z} - \frac{6}{z^2} \cdots - \frac{8}{3}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right]$$

Zeros and Singularities of a function $f(z)$

The points at which a function $f(z)$ takes the value '0' is called zeros of $f(z)$.
In otherwords, a zero is a 'z' at which $f(z)=0$

Eg: (i) $f(z) = (z-1)^2$

at $z=1 \Rightarrow f(z) = (1-1)^2 = 0 \Rightarrow z=1$ is a zero of $f(z) = (z-1)^2$

(ii) $f(z) = z^2 - 1 \Rightarrow f(1) = 1^2 - 1 = 0$
$f(-1) = (-1)^2 - 1 = 0$

$z=1$ and $z=-1$ are the zeros of $f(z) = z^2-1$

(iii) $f(z) = \sin z$.

$\sin z = 0 \Rightarrow z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

There are infinite number of zeros

(iv) $f(z) = e^z$ has no finite zeros.

A function $f(z)$ is singular or has a singularity at a point $z = z_0$, if $f(z)$ is not analytic at $z = z_0$ but every neighborhood of $z = z_0$ contains points at which $f(z)$ is analytic. We also say that $z = z_0$ is a singular point of $f(z)$.

Types of Singular points

1) Isolated singular points : A singular point $z = z_0$ of a function $f(z)$ is called an isolated singular point, if there exist a circle with centre at z_0 which contain no other singular points of $f(z)$.

Eg: (i) $f(z) = \dfrac{z}{z^2-1} = \dfrac{z}{(z+1)(z-1)}$

$z = 1, -1$ are isolated singularities

$f(z) = \dfrac{1}{\sin \pi z}$ $\qquad \sin \pi z = 0$ $\qquad \pi z = n\pi$
$z = \pm n$

$f(z)$ have infinite number of isolated singularities

(2) Poles

If the principal part of Laurent series of $f(z)$ at $z=z_0$ contains only a finite number of terms then the singularity $z=z_0$ is called a pole.

i.e. principal part contains

$$\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_m}{(z-z_0)^m}$$ then $z=z_0$ a pole of order m.

The pole of first order is also known as simple poles.

If $z=z_0$ is a pole of $f(z)$ then

$|f(z)| \longrightarrow \infty$ as $z \longrightarrow z_0$.

(3) Essential Singularity

If the principal part of $f(z)$ contains infinitely many terms then $z=z_0$ is called an essential singularity.

Eg $e^{1/z}$ have an essential singularity at $z=0$.

4) Removable Singularity

A function $f(z)$ have a removable singularity at $z=z_0$. If $f(z)$ is not analytic at $z=z_0$, but can be made analytic there by assigning a suitable value $f(z_0)$

Eg. $\frac{\sin z}{z}$ have a removable singularity at $z=0$.

Problems

1 Determine singularities

1 $f(z) = \tan z$

$= \dfrac{\sin z}{\cos z}$ $\qquad \cos z = 0 \qquad z = (2n+1)\pi/2$

$f(z) = \tan z$ has infinite no. of isolated singularities

2) $f(z) = \dfrac{1}{(z-3)^2(z+5)}$

$(z-3)^2(z+5) = 0 \qquad z = 3$ pole of order 2

$z = -5$ pole of order 1.

3 $e^{1/z}$ $\qquad f(z) = e^{1/z}$

$= 1 + \dfrac{1}{z} + \dfrac{1}{z^2 \, 2!} + \dots$

essential singularity at $z = 0$.

2 Determine zeros.

1 $z^2+25 \qquad z^2+25=0 \Rightarrow z^2=-25 \qquad z = \pm 5i$

$f(5i) = 0 \qquad f'(z) = 2z \qquad f'(5i) = 10i \neq 0$

$f(-5i) = 0 \qquad f'(z) = 2z. \qquad f'(-5i) = -10i \neq 0$

$5i$ and $-5i$ are zeros of order 1.

2 $f(z) = z\tan z \qquad f(z) = 0 \Rightarrow z\tan z = 0$

$z = 0$ & $z = n\pi$

$f'(z) = z\sec^2 z + \tan z \qquad f(n\pi) = 0$

$f'(0) = 0 \qquad f'(n\pi) \neq 0$

$f''(z) = 2z\sec^2 z\tan z + 2\sec^2 z$

$z = n\pi$ of first order

$f''(0) = 2 \neq 0$

$z = 0$ of second order.

Residues

Expand $f(z)$ as Laurent Series and Residue is b_1, i.e. the coefficient of $\frac{1}{z-z_0}$.

Formula for Residues

(1) If $f(z)$ is of the form $\frac{p(z)}{q(z)}$ then Simple pole at z_0 Residue is

$$\underset{z=z_0}{\text{Res}}\, f(z) = \frac{p(z_0)}{q'(z_0)}.$$

(2) $$\underset{z=z_0}{\text{Res}}\, f(z) = \lim_{z \to z_0} (z-z_0) f(z)$$

(3) pole of Order m

$$\underset{z=z_0}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

Residue Theorem

Let $f(z)$ be analytic inside a Simple closed path C and on C, except for finitely many Singular points $z_1, z_2, \ldots z_k$ inside C. Then the integral of $f(z)$ taken Counter Clockwise around C equals $2\pi i$ times the Sum of the residues of $f(z)$ at $z_1, \ldots z_k$

$\oint f(z)\,dz = 2\pi i \sum_{j=1}^{k} \underset{z=z_j}{Res}\, f(z)$

$= 2\pi i\,[\text{Sum of Residues}]$

Pbms

1 Integrate $f(z) = \frac{1}{z^3 - z^4}$ clockwise around the circle $C : |z| = 1/2$

$\rightarrow \oint_C \frac{1}{z^3(1-z)}\,dz$

Here $z = 0$ is a pole of order 3

$z = 1$ is a pole of order 1

$C : |z| = 1/2$ $\quad z = 0 \quad |0| < 1/2$ inside C

$z = 1 \quad |1| > 1/2$ out side C

$R_1 = Res(f(z), z=0) = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} z^3 \frac{1}{z^3(1-z)}$

$= \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \frac{1}{1-z}$

$\frac{d}{dz}(1/(1-z)) = \frac{1}{(1-z)^2}$ $\quad = \frac{1}{2} \lim_{z \to 0} \frac{2}{(1-z)^3} = 1$

$\frac{d}{dz}\frac{1}{(1-z)^2} = \frac{2}{(1-z)^3}$

$\oint_C \frac{1}{z^3 - z^4}\,dz = -2\pi i \times 1 = -2\pi i$

[- Sign indicates C taken in clockwise direction]

2 Find all Singularities and corresponding residues

1) $\frac{\sin 2z}{z^6}$

Ans. $f(z) = \frac{\sin 2z}{z^6}$ Here $z = 0$ pole of order 6

⑥

$$\text{Res}(f(z), z=0) = \frac{1}{(6-1)!} \lim_{z\to 0} \frac{d^5}{dz^5} z^6 \cdot \frac{\sin 2z}{z^6}$$

$$= \frac{1}{5!} \frac{d^5}{dz^6} \sin 2z$$

$$= \frac{1}{5\times4\times3\times2} 32\cos 2z = \frac{4}{15}$$

(2) $\dfrac{z+2}{(z-2)(z+1)^2}$

$z=2$ pole of order 1

$z=-1$ pole of order 2

$$R_1 = \text{Res}(f(z), z=2) = \lim_{z\to 2} (z-2) \frac{z+2}{(z-2)(z+1)^2} = 4/9$$

$$R_2 = \text{Res}(f(z), z=-1) = \frac{1}{1!} \lim_{z\to -1} \frac{d}{dz}(z+1)^2 \cdot \frac{z+2}{(z-2)(z+1)^2}$$

$$= \lim_{z\to -1} \frac{d}{dz} \frac{z+2}{z-2}$$

$$= \lim_{z\to -1} \left\{ \frac{(z-2)-(z+2)}{(z-2)^2} = \frac{-4}{9} \right.$$

(3) $f(z) = \dfrac{e^z}{z^2+4}$

$z^2+4=0 \qquad z = \pm 2i$

$z = 2i, -2i$ are simple poles

$$R_1 = \text{Res}(f(z), z=2i) = \lim_{z\to 2i} (z-2i) \frac{e^z}{(z+2i)(z-2i)}$$

$$= \frac{e^{2i}}{4i}$$

$$R_2 = \text{Re}(f(z), z=-2i) = \lim_{z\to -2i} (z+2i) \frac{e^z}{(z+2i)(z-2i)}$$

$$= \frac{e^{-2i}}{-4i}$$

(3) Evaluate $\oint_C \frac{z-23}{z^2-4z-5}\,dz$ where $C: |z-2-i| = 3.2$

$z^2-4z-5 = (z-5)(z+1)$

$z = 5, -1$ are simple poles

$C: |z-2-i| = 3.2$

$z=5 \quad |5-2-i| = |3-i| = \sqrt{9+1} = \sqrt{10} < 3.2$ inside c

$z=-1 \quad |-1-2-i| = |-3-i| = \sqrt{9+1} = \sqrt{10}$ inside c

$$R_1 = \text{Res}(f(z), z=5) = \lim_{z\to 5} (z-5)\frac{z-23}{(z-5)(z+1)}$$

$$= \lim_{z\to 5} \frac{z-23}{z+1} = -3$$

$$R_2 = \text{Res}(f(z), z=-1) = \lim_{z\to -1} (z+1)\frac{z-23}{(z-5)(z+1)}$$

$$= \lim_{z\to -1} \frac{z-23}{z-5} = 4$$

By Residue thm

$$\oint_C \frac{z-23}{z^2-4z-5}\,dz = 2\pi i[\text{sum of residues}]$$

$$= 2\pi i[-3+4] = \underline{2\pi i}$$

(4) Use Cauchy Residue thm to evaluate $\oint_C \frac{z\,dz}{(z-1)(z-2)^2}$ where C is the circle $|z-2| = 1/2$.

Ans $f(z) = \frac{z}{(z-1)(z-2)^2}$ Here $z=1$ pole of order 1, $z=2$ pole of order 2

$C. |z-2| = 1/2 \quad z=1 \quad |1-2| = 1 > 1/2$ outside C

$z=2 \quad |2-2| = 0 < 1/2$ inside C

$$R_1 = \text{Res}(f(z), z=2) = \frac{1}{1!}\lim_{z\to 2}\frac{d}{dz}(z-2)^2\frac{z}{(z-1)(z-2)^2}$$

$$= \lim_{z\to 2}\frac{d}{dz}\frac{z}{z-1} = \lim_{z\to 2}\left\{\frac{(z-1)-z}{(z-1)^2}\right\}$$

$$= \underline{\underline{-1}}$$

$\therefore$ By Residue thm $\oint_C \frac{z}{(z-1)(z-2)^2}\,dz = 2\pi i\,(\text{sum of residues})$

$= 2\pi i \times -1$

$= -2\pi i$ //

(5) Evaluate $\int_C \frac{\sin z}{(z-1)^2(z^2+9)}$ where C is the circle (7)

$|z-3i| = 1$

Ans

$$f(z) = \frac{\sin z}{(z-1)^2(z^2+9)} = \frac{\sin z}{(z-1)^2(z+3i)(z-3i)}$$

Here $z=1$ pole of order 2

$z=3i$ pole of order 1

$z=-3i$ pole of order 1

$C: |z-3i|=1$

$z=1$ $|1-3i| = \sqrt{1^2+(-3)^2} = \sqrt{10} > 1$ outside C

$z=3i$ $|3i-3i| = 0 < 1$ inside C.

$z=-3i$ $|-3i-3i| = |-6i| > 1$ outside C.

$$R_1 = \operatorname{Res}(f(z), z=3i) = \lim_{z\to 3i} (z-3i)\, f(z)$$

$$= \lim_{z\to 3i} (z-3i) \cdot \frac{\sin z}{(z-1)^2(z+3i)(z-3i)}$$

$$= \lim_{z\to 3i} \frac{\sin 3i}{(3i-1)^2 \cdot 6i} = \frac{i \sinh 3}{[-9-6i+1]\cdot 6i}$$

$$= \frac{\sinh 3}{(-8-6i)6}$$

$$= \frac{(8-6i)\sinh 3}{(8-6i)(-8-6i)6} = \frac{(8-6i)\sinh 3}{(36-64)6}$$

$$= \frac{(4-3i)\sinh 3}{-300}$$

$$= -\frac{(4-3i)\sinh 3}{300}$$

$$\oint_C \frac{\sin z}{(z-1)^2(z^2+9)} = \frac{2\pi i \times -(4-3i)\sinh 3}{300}$$

$$= \frac{\pi(-3-4i)\sinh 3}{150}$$

6 Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}$ where C is the circle $|z+i| = 2$

Ans

$$f(z) = \frac{z-1}{(z+1)^2(z-2)}$$

$z = -1$ pole of order 2

$z = 2$ pole of order 1

$C: |z+i| = 2$

$z = -1$ $\quad |-1+i| = \sqrt{2} < 2$ inside c.

$z = 2$ $\quad |2+i| = \sqrt{5} > 2$ outside c

$$R_1 = \operatorname{Res}(f(z), z=-1) = \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz} (z+1)^2 \cdot f(z)$$

$$= \lim_{z \to -1} \frac{d}{dz} (z+1)^2 \cdot \frac{z-1}{(z-2)(z+1)^2}$$

$$= \lim_{z \to -1} \frac{(z-2) - (z-1)}{(z-2)^2} = -1/9$$

By residue thm

$$\oint_C \frac{z-1}{(z+1)^2(z-2)} dz = 2\pi i \times -1/9 = \frac{-2\pi i}{9}$$

7. Evaluate $\int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)^2(z+2)} dz$ where C is $|z| = 3$

Ans

$$f(z) = \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)^2(z+2)}$$

$z = -1$ pole of order 2

$z = -2$ pole of order 1

$C: |z| = 3$

$z = -1$ $\quad |-1| < 3$ inside c

$z = -2$ $\quad |-2| < 3$ inside c.

$$R_1 = \operatorname{Res}(f(z), z=-1) = \frac{1}{1!} \lim_{z \to -1} (z+1)^2 \cdot \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)^2 (z+2)} \quad (8)$$

$$= \lim_{z \to -1} \frac{(z+2)\{-\sin \pi z^2 \times 2\pi z + \cos \pi z^2 \times 2\pi z\} - \{\cos \pi z^2 + \sin \pi z^2\}}{(z+2)^2}$$

$$= \frac{(-1+2)\{[-\sin \pi \times -2\pi + \cos \pi \times -2\pi] - \{\cos \pi + \sin \pi\}}{(-1+2)^2}$$

$$= \frac{2\pi+1}{1^2} = \underline{2\pi+1}$$

$$R_2 = \operatorname{Res}(f(z), z=-2) = \lim_{z \to -2} (z+2) \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)^2 (z+2)}$$

$$= \frac{\cos 4\pi + \sin 4\pi}{(-2+1)^2} = \underline{1}$$

$$\oint_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)^2 (z+2)} = 2\pi i \{2\pi + 1 + 1\} = \underline{4\pi i (\pi+1)}$$

Residue Integration of real integrals

Integral of the type $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

Put $e^{i\theta} = z$ then $\cos\theta = \frac{z+\bar{z}}{2} = \frac{z+\frac{1}{z}}{2}$

$\sin\theta = \frac{z-\bar{z}}{2i} = \frac{1}{2i}\left[z - \frac{1}{z}\right]$

$d\theta = \frac{dz}{iz}$

$\therefore$ Integral become $\int_C f(z) \frac{dz}{iz}$

Pbm

1 Evaluate $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$ $a > b > 0$ using contour integration.

A. Let $z = e^{i\theta}$ $|z| = 1$

$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) \quad d\theta = \frac{dz}{iz}$

$a+b\cos\theta = a+\frac{b}{2}\left(z+\frac{1}{z}\right) = a+\frac{b}{2z}(z^2+1)$

$= \frac{2az+bz^2+b}{2z}$

$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{\frac{dz}{iz}}{\frac{2az+bz^2+b}{2z}}$

$= \frac{2}{i}\int_C \frac{dz}{2az+bz^2+b}$

$= \frac{2}{i}\int_C f(z)\,dz$

$f(z) = \frac{1}{bz^2+2az+b}$

$bz^2+2az+b=0 \Rightarrow z = \frac{-2a\pm\sqrt{4a^2-4b^2}}{2b}$

$z = \frac{-a\pm\sqrt{a^2-b^2}}{b}$

$f(z) = \frac{1}{b\left[z-\left[\frac{-a+\sqrt{a^2-b^2}}{b}\right]\right]\left[z-\left[\frac{-a-\sqrt{a^2-b^2}}{b}\right]\right]}$

$z = \frac{-a+\sqrt{a^2-b^2}}{b}, \frac{-a-\sqrt{a^2-b^2}}{b}$ are simple poles

$C: |z|=1 \quad \left|\frac{-a+\sqrt{a^2-b^2}}{b}\right| = \left|-\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}\right| < 1$ inside C since $a/b>1$

$\left|\frac{-a-\sqrt{a^2-b^2}}{b}\right| = \left|-\frac{a}{b}-\sqrt{\frac{a^2}{b^2}-1}\right| > 1$ outside C

$R_1 = \mathrm{Res}\left(f(z), z=\frac{-a+\sqrt{a^2-b^2}}{b}\right) = \lim_{z\to\frac{-a+\sqrt{a^2-b^2}}{b}} \left[z-\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)\right] \times \frac{1}{b\left[z-\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)\right]\left[z-\left[\frac{-a-\sqrt{a^2-b^2}}{b}\right]\right]}$

$= \frac{1}{b\left[\frac{-a+\sqrt{a^2-b^2}}{b}-\left[\frac{-a-\sqrt{a^2-b^2}}{b}\right]\right]} = \frac{1}{b\times 2\frac{\sqrt{a^2-b^2}}{b}}$

$= \frac{1}{}$

$\int f(z)dz = 2\pi i R_1$ (9)

$= 2\pi i \times \frac{1}{2\sqrt{a^2-b^2}} = \frac{\pi i}{\sqrt{a^2-b^2}}$

$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{i}\int f(z)dz$

$= \frac{2}{i} \times \frac{\pi i}{\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$

Q. Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ using contour integration.

Ans

$z = e^{i\theta}$ $\quad d\theta = \frac{dz}{iz}$ $\quad \cos\theta = \frac{1}{2}(z + 1/z)$

$2+\cos\theta = 2 + \frac{1}{2}(z + 1/z)$

$= 2 + \frac{z^2+1}{2z} = \frac{z^2+4z+1}{2z}$

$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_C \frac{dz/iz}{\frac{z^2+4z+1}{2z}} = \frac{2}{i}\int \frac{dz}{z^2+4z+1}$ $\rightarrow (1)$

$f(z) = \frac{1}{z^2+4z+1}$

$z = -2 \pm \sqrt{3}$ are singular pts of order 1

$f(z) = \frac{1}{[z-(-2+\sqrt{3})][z-(-2-\sqrt{3})]}$

$C: |z| = 1$ $\quad |-2+\sqrt{3}| < 1$ inside c

$|-2-\sqrt{3}| > 1$ outside c

$R_1 = Res(f(z), z = -2+\sqrt{3}) = \lim_{z \to -2+\sqrt{3}} [z-(-2+\sqrt{3})] \frac{1}{[z-(-2+\sqrt{3})][z-(-2-\sqrt{3})]}$

$= \frac{1}{\ldots} = \frac{1}{2\sqrt{3}}$

By Residue thm

$$\oint f(z)\,dz = 2\pi i \times \frac{1}{2\sqrt{3}} = \frac{\pi i}{\sqrt{3}}$$

By (1) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i} \times \frac{\pi i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$

(3) Show that $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \pi/6$

Ans $z = e^{i\theta}$ $\quad |z| = 1$ $\quad d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}(z + 1/z) = \frac{z^2+1}{2z}$ $\quad \cos 2\theta = \frac{1}{2}(z^2 + 1/z^2)$

$= \frac{1}{2}\frac{z^4+1}{z^2}$

$$\frac{\cos 2\theta}{5+4\cos\theta} = \frac{\frac{1}{2}\frac{(z^4+1)}{z^2}}{5+4\times\frac{(z^2+1)}{2z}} = \frac{\frac{z^4+1}{2z^2}}{\frac{5z+2z^2+2}{z}} = \frac{z^4+1}{2z[2z^2+5z+2]}$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \int_C \frac{z^4+1}{2z(2z^2+5z+2)}\frac{dz}{iz}$$

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2(2z^2+5z+2)}\,dz$$

$$= \frac{1}{2i}\int_C f(z)\,dz$$

$$f(z) = \frac{1}{z^2(2z^2+5z+2)}$$

$$= \frac{1}{2z^2[z^2+\frac{5}{2}z+1)}$$

$z^2+5/2\,z+1 = 0 \Rightarrow z = -1/2, -2$

$z=0$ pole of order 2

$z=-1/2$ pole of order 1

$z=-2$ pole of order 1

$C: |z|=1 \Rightarrow z=0,\ z=-1/2$ lies inside c and $z=-2$ outside c

$R_1 = \text{Res}\,(f(z), z=0)$

$= \frac{1}{1!} \lim_{z\to 0} \frac{d}{dz} z^2 \cdot \frac{z^4+1}{2z^2(z^2+5/2\,z+1)}$

$= \lim_{z\to 0} \frac{(z^2+\frac{5}{2}z+1)\,4z^3 - (z^4+1)(2z+5/2)}{(z^2+5/2\,z+1)^2} = -5/4$

$R_2 = \text{Res}\,(f(z), z=-1/2)$

$= \lim_{z\to -1/2} (z+1/2) \frac{z^4+1}{2z^2(z+1/2)(z+2)} = \frac{17}{12}$

By Residue thm

$\oint_C f(z)\,dz = 2\pi i \times \text{sum of residues}$

$= 2\pi i \left[-\frac{5}{4} + \frac{17}{12} \right] = \frac{\pi i}{3}$

$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \frac{1}{2i} \int f(z)\,dz$

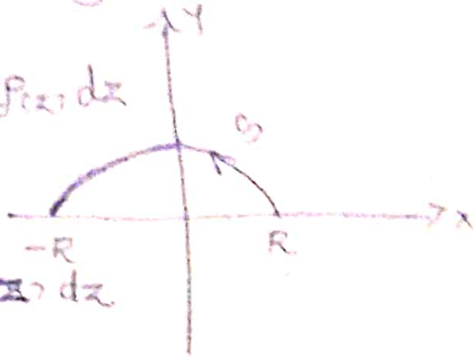
$= \frac{1}{2i} \cdot \frac{\pi i}{3} = \pi/6$

Integral of the type $\int_{-\infty}^{\infty} f(x)\,dx$

$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$

$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_S f(z)\,dz$

$2\pi i \sum \text{Res}\, f(z) = \int_{-R}^{R} f(x)\,dx + \int_S f(z)\,dz$



$\int_{-R}^{R} f(x)\,dx = 2\pi i \sum \text{Res} f(z) - \int_S f(z)\,dz$

$\lim_{R\to\infty} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{Res}\, f(z)$

If poles are on real axis

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{Res } f(z) + \pi i \sum \text{Res } f(z)$$

residues corresponding pts in upper half plane — corresponding to real axis

Pbms

1 Evaluate $\int_0^{\infty} \frac{dx}{x^2+1}$

A: Consider $\int f(z)\,dz = \int_C \frac{dz}{z^2+1}$

Where C is the upper semi circle $|z| = R$

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_1} f(z)\,dz \quad \text{---(1)}$$

$f(z) = \frac{1}{z^2+1}$ $\quad z^2+1=0 \Rightarrow z = \pm i$ are simple poles.

$z = i$ inside C

$z = -i$ outside C

$$R_1 = \text{Res}\,(f(z), z=i) = \lim_{z\to i} (z-i)\frac{1}{(z+i)(z-i)} = \frac{1}{2i}$$

$\lim_{R\to\infty} \int_C f(z)\,dz = \int_{-\infty}^{\infty} (f(x))$ $2\pi i$ (sum of residues

$$\int_C f(z)\,dz = 2\pi i \times \frac{1}{2i} = \pi$$

$$(1) \Rightarrow \pi = \int_{-R}^{R} f(x)\,dx + \int_{C_1} f(z)\,dz$$

$\left\{ \lim_{R\to\infty} \int_{C_1} f(z)\,dz = 0 \right.$

$R \to \infty$

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi$$

$$2\int_0^{\infty} \frac{1}{x^2+1}\,dx = \pi \quad \left\{ f(x) = \frac{1}{x^2+1} \text{ even} \right)$$

$$\int_0^{\infty} \frac{1}{x^2+1}\,dx = \pi/2$$

Q Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)} dx$

Ans $\int_C f(z)dz = \int \frac{z^2}{(z^2+9)(z^2+4)} dz$

$$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_1} f(z)dz$$

$R \to \infty$ $\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx$

$$f(z) = \frac{z^2}{(z^2+9)(z^2+4)}$$

$z = \pm 3i, \pm 2i$ are simple poles

$z = 3i, 2i$ inside C.

$R_1 = Res(f(z), z=3i) = \lim_{z \to 3i} \frac{(z-3i)\, z^2}{(z+3i)(z-3i)(z^2+4)}$

$= \frac{-9}{6i \times -5} = \underline{\frac{3}{10i}}$

$R_2 = Res(f(z), z=2i) = \lim_{z \to 2i} \frac{(z-2i)\, z^2}{(z^2+9)(z+2i)(z-2i)}$

$= \frac{-4}{5 \times 4i} = \underline{-\frac{1}{5i}}$

$\int_C f(z)dz = 2\pi i$ (Sum of residues)

$= 2\pi i\left(\frac{3}{10i} - \frac{1}{5i}\right) = \frac{2\pi i}{10i}[3-2] = \underline{\frac{\pi}{5}}$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)} dx = \underline{\frac{\pi}{5}}$$

(3) $\int_0^{\infty} \frac{dx}{x^2+4}$ using Contour integration

Ans $\int_C f(z)dz = \int_C \frac{dz}{z^2+4}$

$\int f(z)dz = \int^{R} f(x)dx + \int f(z)dz$

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} f(x)\,dx = \int_C f(z)\,dz \longrightarrow (1)$$

$$f(z) = \frac{1}{z^2+4}$$

$z^2+4=0$ $\quad z = \pm 2i$ are simple poles

$z = 2i$ inside C $\quad z = -2i$ outside C.

$$R_1 = \text{Res}(f(z), z=2i) = \lim_{z\to 2i} (z-2i)\frac{1}{(z+2i)(z-2i)} = \frac{1}{4i}$$

$$\int_C f(z)\,dz = 2\pi i\,(\text{sum of residues}) = 2\pi i \times \frac{1}{4i} = \underline{\underline{\pi/2}}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \underline{\underline{\pi/2}}$$

$$2\int_0^{\infty} f(x)\,dx = \pi/2 \implies \int_0^{\infty} f(x)\,dx = \underline{\underline{\pi/4}}$$

$\int^{\infty} dx$